Modeling Motion of the Lissajous Sand Pendulum

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Report Submitted: April 19, 2019

Contents

1 Abstract

The Lissajous Sand Pendulum is a mass-variant, double pendulum which produces different paths of motion dependent upon the pendulum's initial release point. Lissajous pendulums differ from standard double pendulums because they eject mass in the form of a fluid or loose solid such as sand. This ejection of mass creates a rocket-like propulsion effect for the pendulum when the fluid mass leaving the system is equal to or more than the mass remaining inside the pendulum bob. As a result of the pendulum's time dependent mass and coupled motion, the pendulum exhibits not one distinct set of motion but several curve sets belonging to what mathematicians call a Family of Lissajous Curves. My research aims to study and classify the motion of the Lissajous Pendulum by analyzing the trajectories and energy loss resulting from different initial conditions. This project seeks to identify an energy damping coefficient that accurately models all studied curve sets. Preliminary conclusions indicates that each initial condition retains a significantly different damping coefficient. Further, the classification of each curve set's damping coefficient may indicate a mathematical relationship between the shortening and elongating of different pendulum paths providing more insight into modeling the motion of the Lissajous Pendulum.

2 Introduction

With the rise of technology and the popularization of science fiction, there has also been a rise in the demand for designs or branding that indicates a company or lab is future thinking. The design pattern that has commonly been branded as futuristic in Star Trek, Alien, and many other science fiction genres are Lissajous Curve patterns. While these patterns are typically displayed on an oscilloscope or are used on a monitor to indicate communication broadcasting, they have served the purpose of logos for MIT's Lincoln Lab, Disney's Movies Anywhere streaming service, and the Australian governments Australian Broadcasting Corporation. This project seeks to understand the formation of Lissajous trace curves by studying and classifying the energy damping of the Lissajous sand pendulum in turn identifying how the system changes based on its initial parameters.

The Lissajous sand pendulum is a combination of a leaky pendulum and a spherical pendulum. A leaky pendulum is a simple pendulum system in which the mass of the pendulum bob leaks out (Arun). This causes the center of mass to move upwards which then reduces the moment of inertia. A second result of this shift in the center of mass causes the suspension of the pendulum becomes more dominant in the mass distribution because the center of mass shifts upwards along the suspension. For simplicity's sake, if we assume that the moment of inertia of the leaky pendulum when taut is "rigid" then we can model the change in moment of inertia as the follow (equation 1).

$$I = \frac{1}{3}M_{rope}L^2 + m_0 \times (1-\alpha)L^2 \tag{1}$$

Where the moment of inertia is modeled as a rigid rod around a fixed end, m_0 is the original mass inside the pendulum bob, α is the percentage of mass that has left the pendulum bob, and L is the length of the pendulum we take this to be the moment of inertia of our leaky pendulum. This is the formula that we use in our simulation.

The second similar behavior is that of the spherical pendulum. The spherical pendulum exhibits periodic behavior that can be modeled as a reworked to vary in time due to mass loss (equation 2).

$$T = \frac{T_0}{\sqrt{1 - \frac{L}{4g}(\frac{1}{m(t)}\frac{dm}{dt})^2}} \tag{2}$$

We will return to this relation in coding the simulation substituting the current mass with the moment of inertia and the derivative of mass over time as a time function of α where α is the percentage of mass lost per second given the equation below.

$$m(t) = M_{rope} + m_0(1 - \alpha t) \tag{3}$$

In this study, the goal was to identify and classify the energy damping coefficient of different Lissajous curve models utilizing physical and simulated models. The energy loss was found by analyzing the periodic amplitude decay.

$$E = \epsilon k A^2 \tag{4}$$

Where ϵ represents the amplitude decay between periods, k is a constant and A is the amplitude of our pendulum's trace curves. This can be expanded upon by writing the amplitude as a function of the oscillation. We represent the n^{th} amplitude as an exponential decay related to the ratio of the current oscillation T_n and the first or original oscillation T_0 as shown in equation five.

$$A_n = A_0 e^{(1 - \frac{T_n}{T_0})} \tag{5}$$

Combining equation 4 and 5 we arrive at the final model for analyzing our energy damping as:

$$A_1 e^{(1-\frac{T_{n+1}}{T_0})} = \epsilon k \times A_0 e^{(1-\frac{T_n}{T_0})} \tag{6}$$

$$\epsilon = (\frac{A_1}{kA_0})^2 \times e^{\frac{1-\frac{T_{n+1}}{T_0}}{1-\frac{T_n}{T_0}}} \approx (\frac{A_1}{A_0})^2 \times \frac{1}{k} e^{\frac{T_n}{T_0}-1}$$

$$\epsilon = (\frac{A_1}{A_0})^2 \times Ce^{\frac{T_n}{T_0}-1} \tag{7}$$

Note that $(\frac{T_n}{T_0} \leq 1)$ holds true because the system decays through time thus the derivation identifies how to find the energy damping coefficient based upon the amplitude decay of the system.

3 Experimental Methods

3.1 Curve Deprecation

One of the methods of obtaining data from the Lissajous sand Pendulum consisted of utilizing amplitude decay between similar curves. As the pendulum traces out curves by dispensing sand, it becomes apparent to an observer that the curves are the repeated trace of the prior curve with a lower amplitude. This decrease in amplitude is a result of energy loss and can be measured. This method of data gathering requires measuring the distance between the

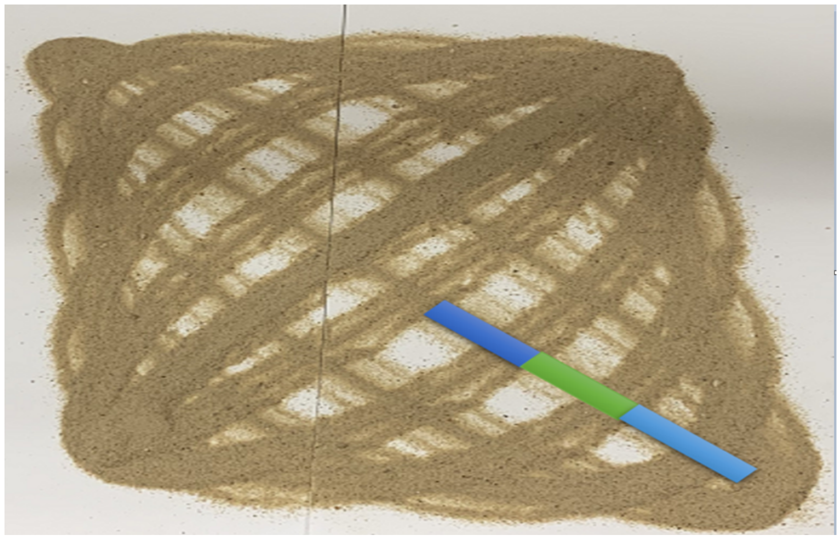

Figure 1: Curve Deprecation Example

apex of two curves. Shown in figure 1, the curves are traced from the exterior inwards, neglecting the first curve to remove error from the experimenter adding energy to the system. The data gathered from this section is used to identify the difference in amplitude between curve sets. The ratio of a curve and its preceding curve provide insight into how much energy is lost as the pendulum decayed inward.

3.2 Origin Line Tracing

The other method used to gather data was Origin Line Tracing. This method measures the distance from the center image to each curve apex (figure 2).

This provides the distance from curve amplitudes with a reference of zero. Obtaining a periodic amplitude measurement that approaches zero is useful in learning more about how the pendulum is losing energy, exponentially or linearly. If data from only the prior method, curve deprecation, was used it becomes difficult to discern whether there is a linear trend or an exponential trend because the data consists of relative measurements. This method is used to provide a zero point in amplitude measurement providing a base of measurement to cross-analyze data with both methods.

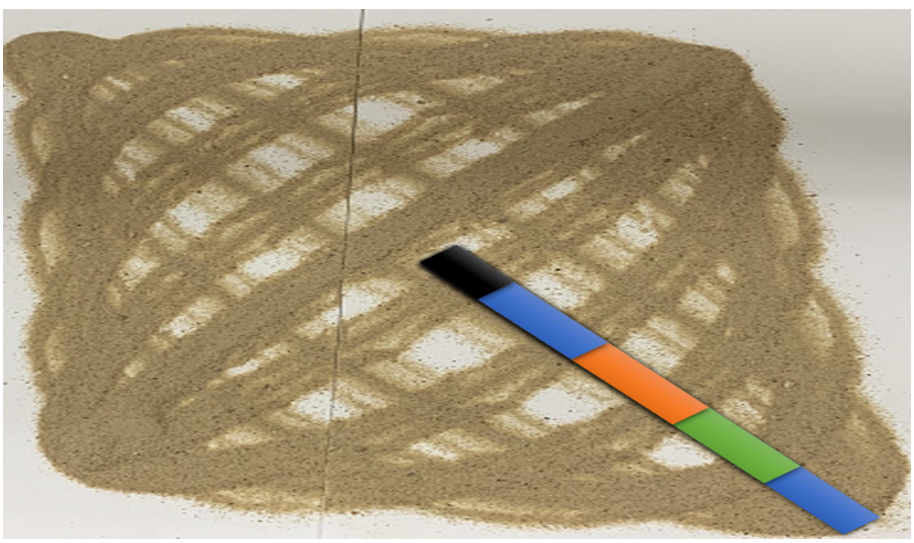

Figure 2: Origin Line Tracing Example

3.3 Simulation Comparison

The verification method of this experiment is to utilize the results of the physical sand pendulum and contrast them with the idealistic curve traces. In order to create ideal curve traces, I developed a script in Python to produce the curves. This method enhances the verification of the physical curve traces by utilizing a variable energy damping coefficient that can be adjusted until the simulated traces share curve amplitude deprecation ratios identical to the physical experiment. Once these values are matched, the energy coefficient parameter models the energy damping coefficient for that trace curve model.

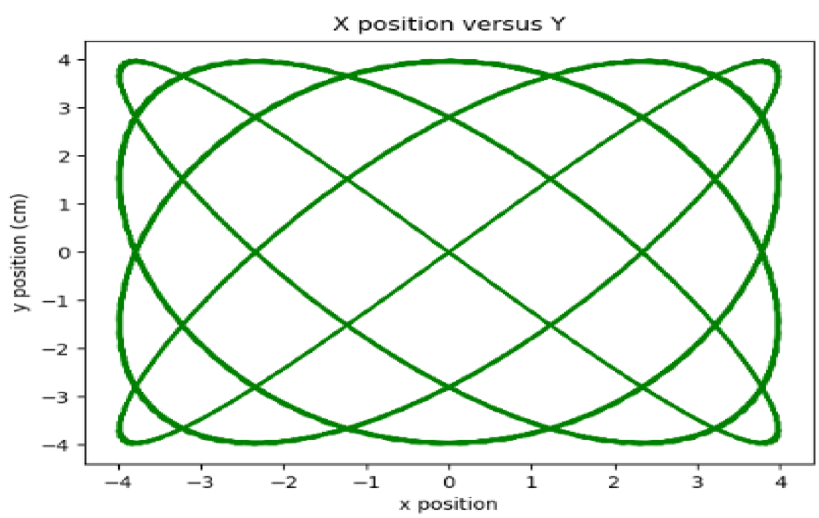

Figure 3: Ideal Simulated Lissajous Curve

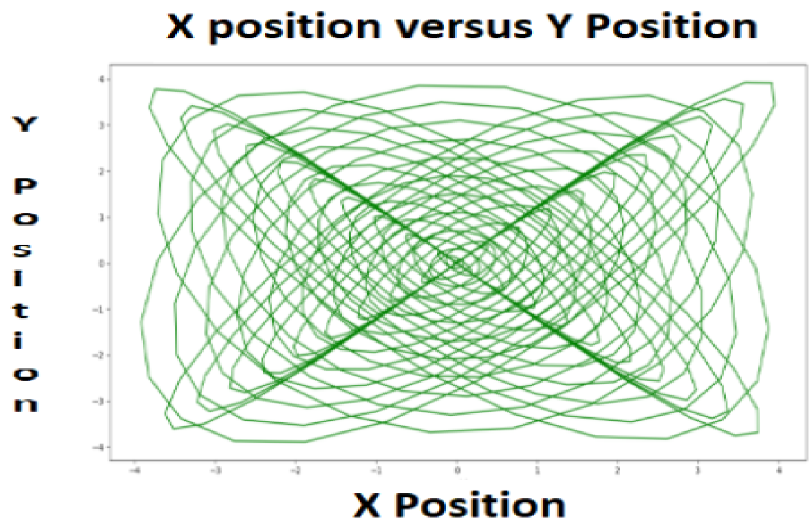

Figure 4: Simulated Lissajous Curve with Energy Damping

4 Data

Curve Set	**Damping Coefficient**	**First Deprecation**	**Second Deprecation**	**Origin Measurement [mm]**
1	0.470	0.743	0.631	34.5±0.5
2	0.505	0.772	0.654	31.75±0.5
3	0.747	0.891	0.838	42.1±0.5
4	0.631	0.829	0.745	51.7±0.5
5	0.453	0.497	0.407	33.7±0.5
6	0.465	0.520	0.451	34.1±0.5

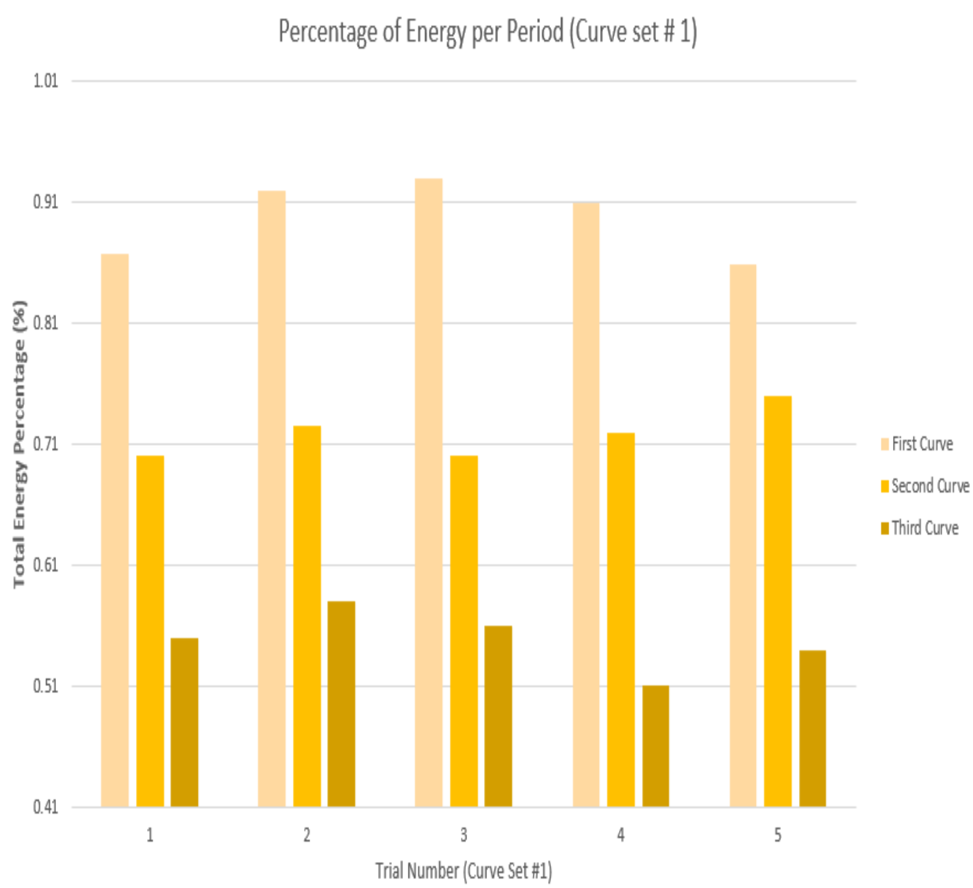

Figure 5: Graphical representation of curve set #1 total energy loss per period.

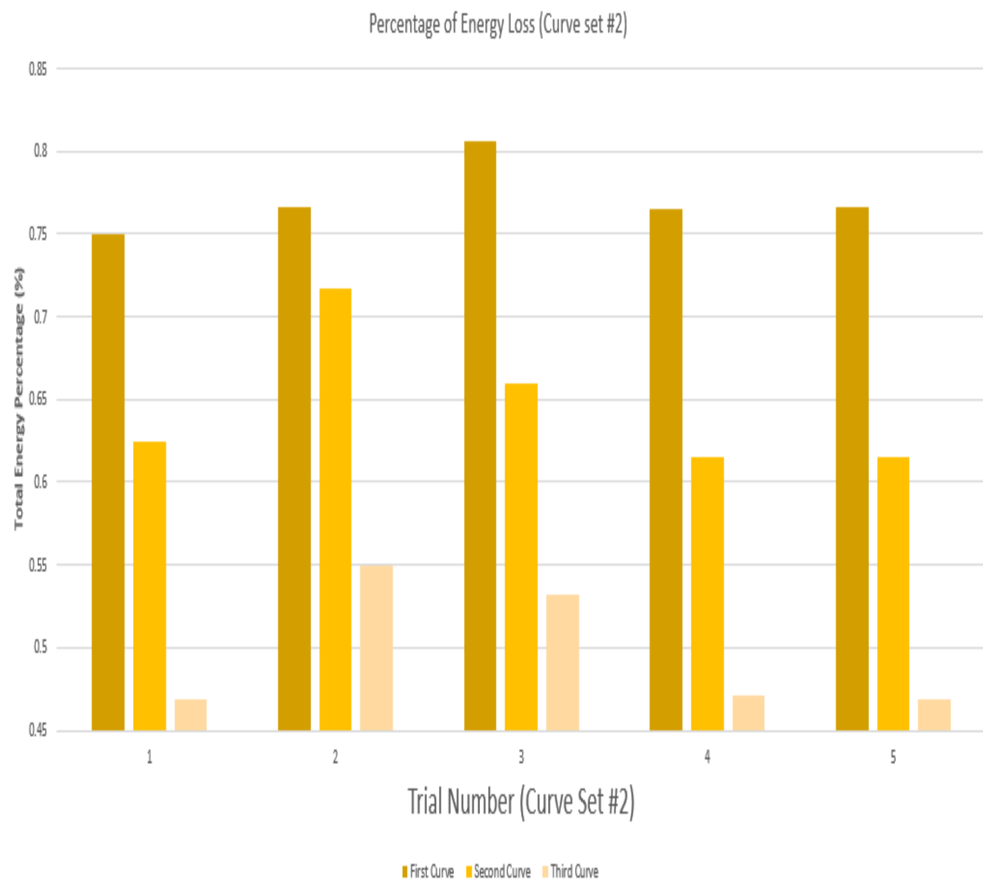

Figure 6: Graphical representation of curve set #2 total energy loss per period.

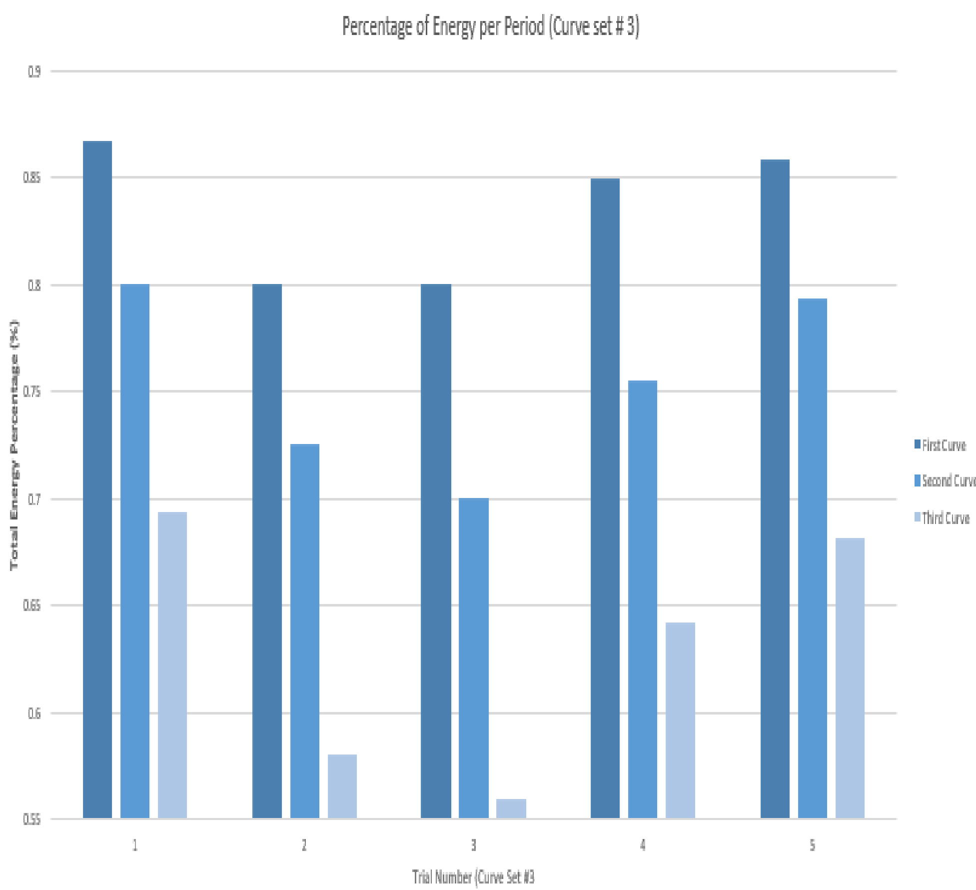

Figure 7: Graphical representation of curve set #3 total energy loss per period.

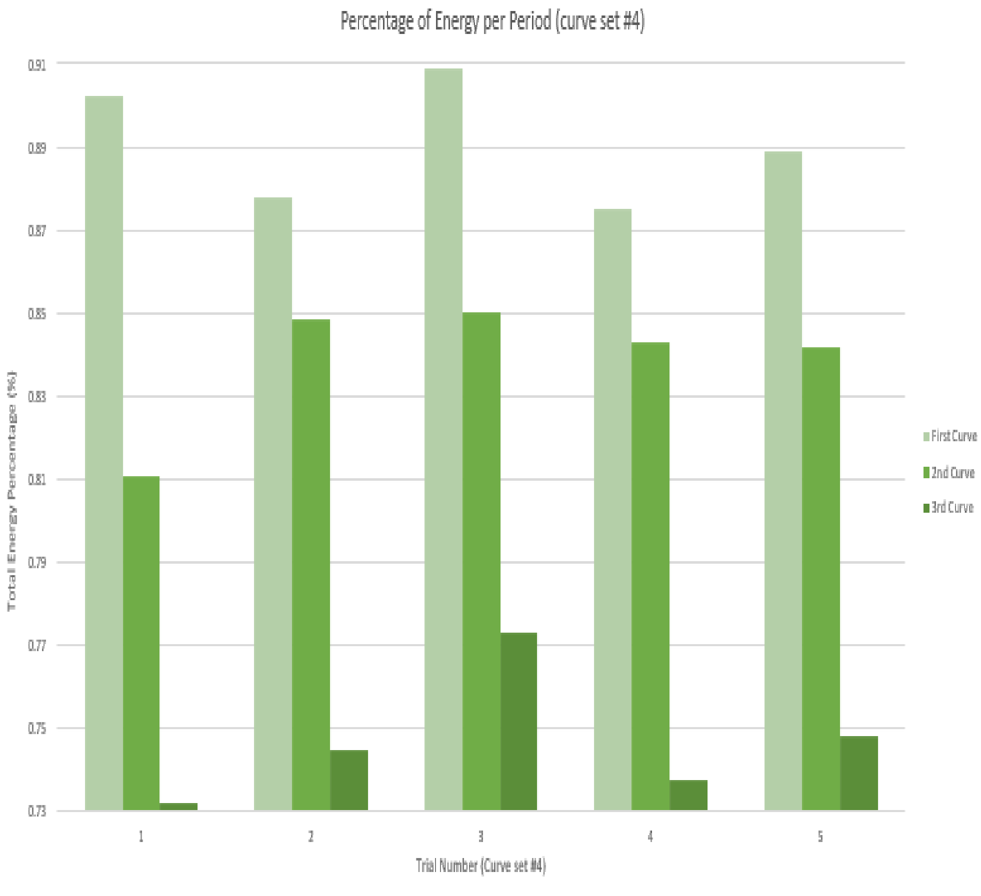

Figure 8: Graphical representation of curve set #4 total energy loss per period.

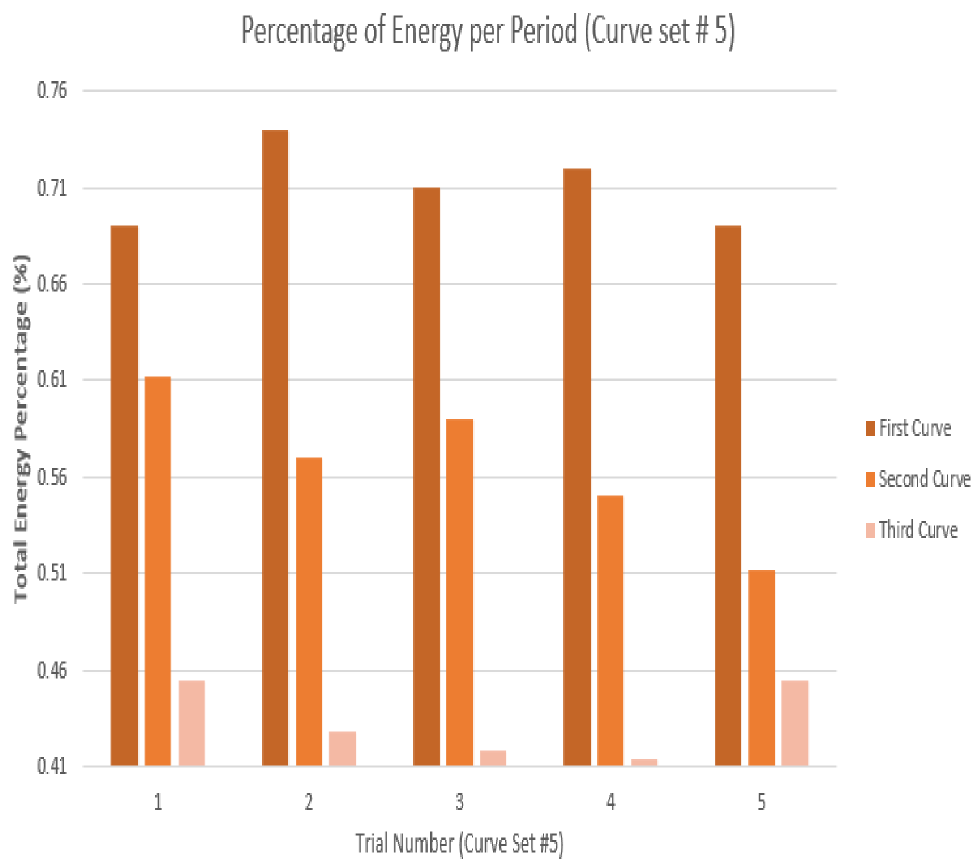

Figure 9: Graphical representation of curve set #5 total energy loss per period.

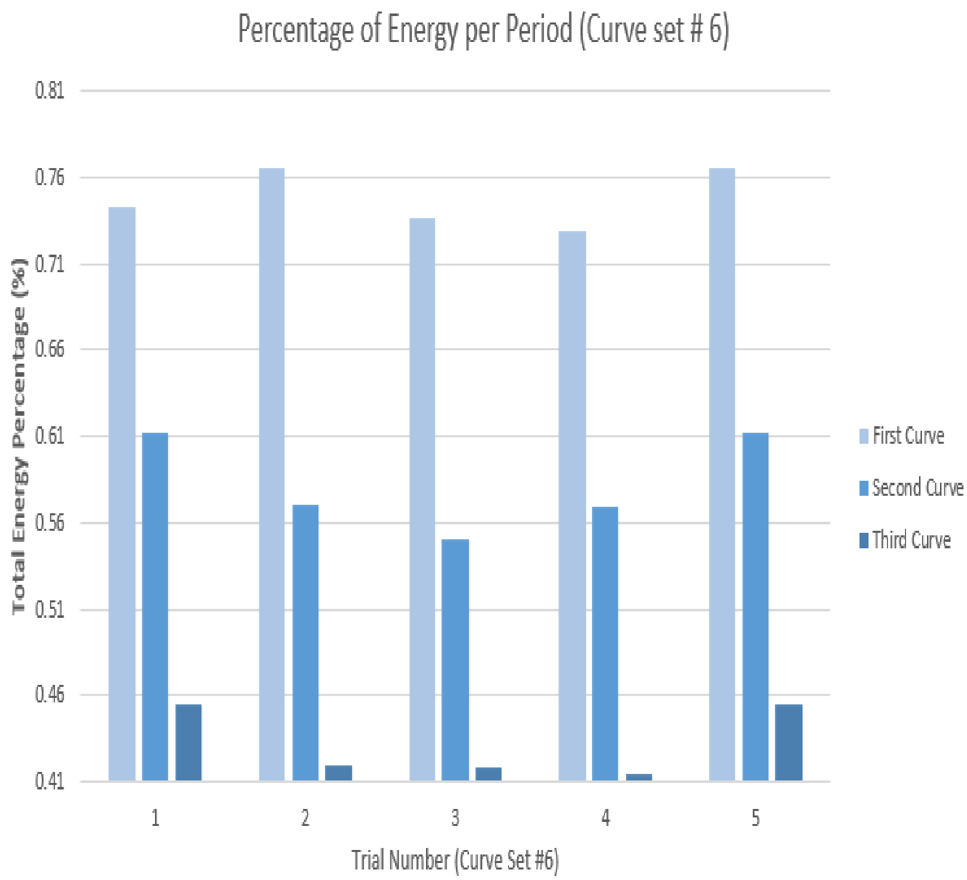

Figure 10: Graphical representation of curve set #6 total energy loss per period.

5 Conclusions

Preliminary data indicated that energy loss decays exponential within each curve set. After further analysis, the relationship between curve set decay became linear as periods shortened and longer periods tended towards exponential decay. In addition to this observation, the identified trend classified larger trace patterns as exhibiting longer initial periods of motion and consequently exhibiting the largest energy damping effects. While discrepancies in data existed, error-ridden data was corrected for by implementing the origin-line tracing method.

Further experiments that could utilize or enhance these findings consists of- using pendulum's with different flow rates, measuring curve sets with different height intervals, or swapping the sand pendulum out for LED's and perform a light tracing experiment of a Lissajous pendulum. All of the above suggestions are different methods that can reveal new aspects of Lissajous curves.

6 Bibliography

- The Motion of the Spherical Pendulum Subjected to a D_n Symmetric Perturbation. Pascal Chossat Nawaf M. Bou-Rabee

- The Moving Center of Mass of a Leaking Bob. P. Arun

7 Appendix

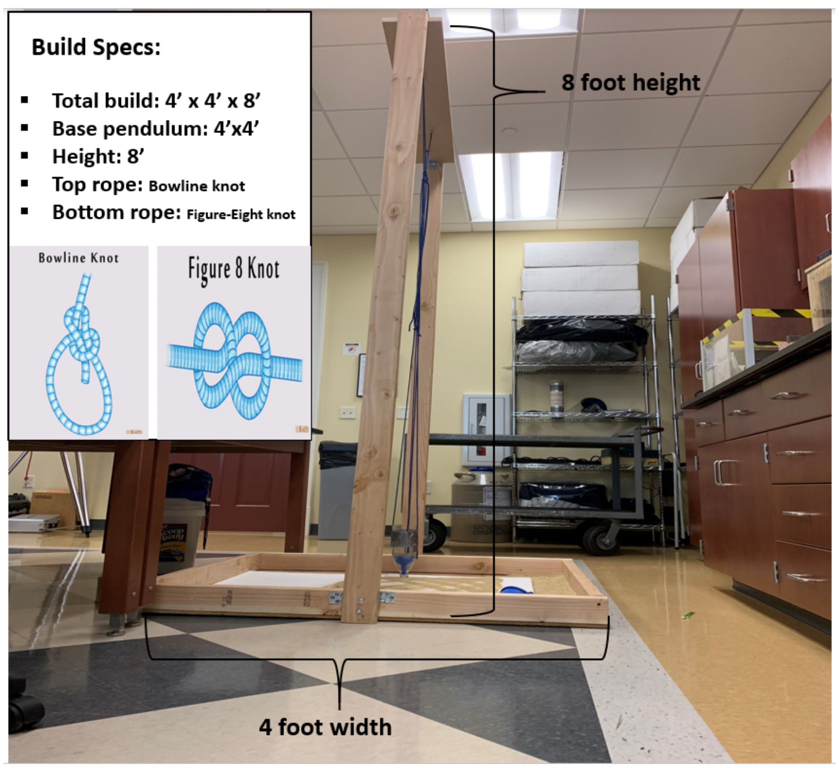

Figure 11: Full Pendulum with dimensions and knot description

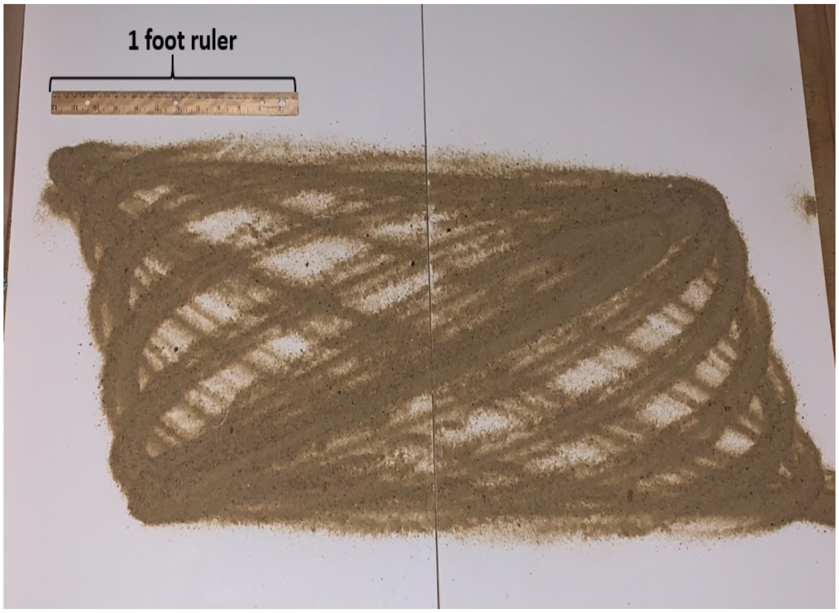

Figure 12: Curve Set 1

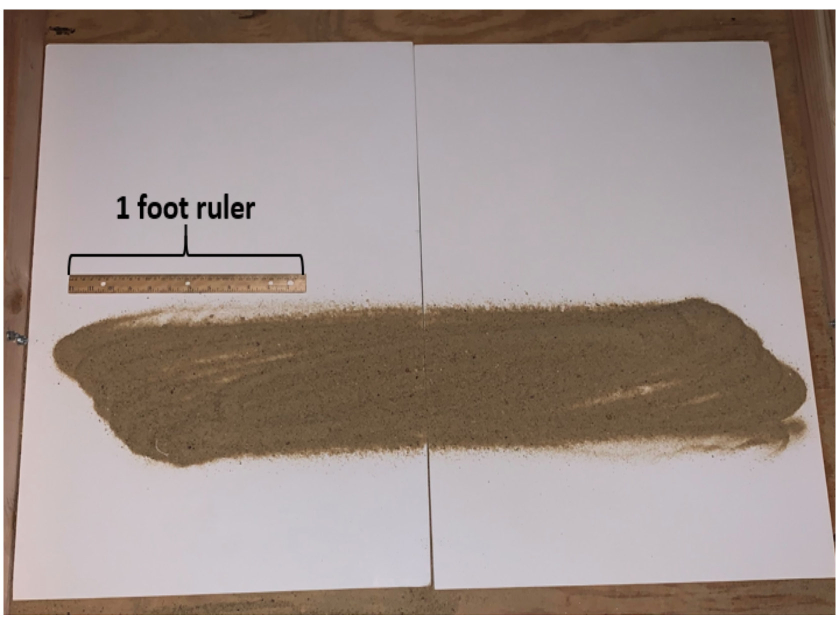

Figure 13: Curve Set 2

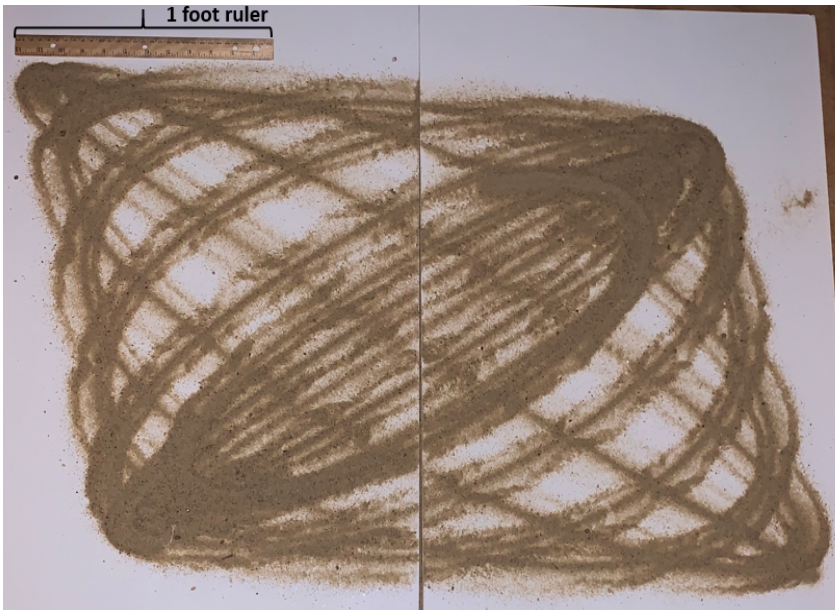

Figure 14: Curve Set 3

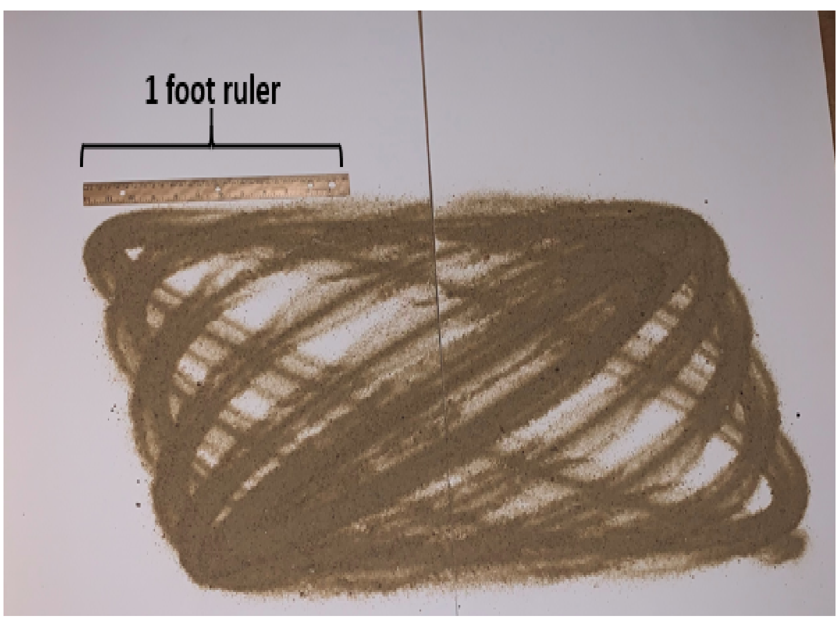

Figure 15: Curve Set 4

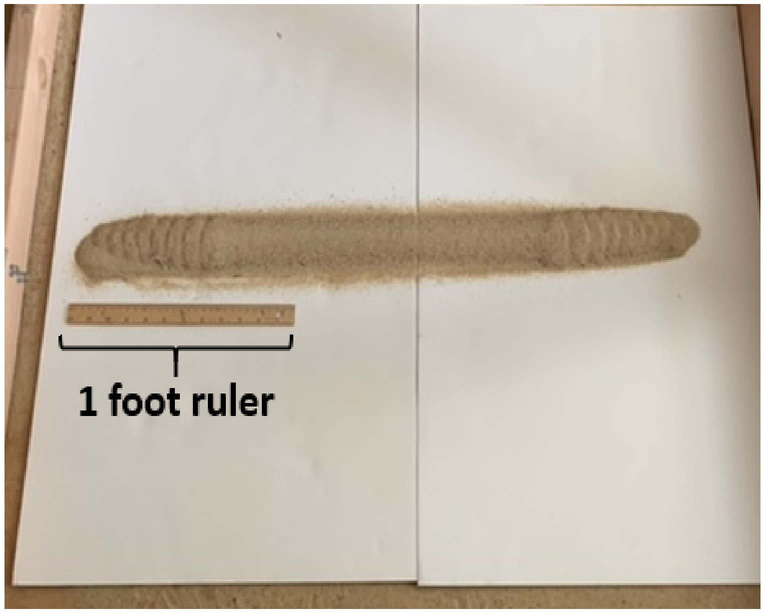

Figure 16: Curve Set 5

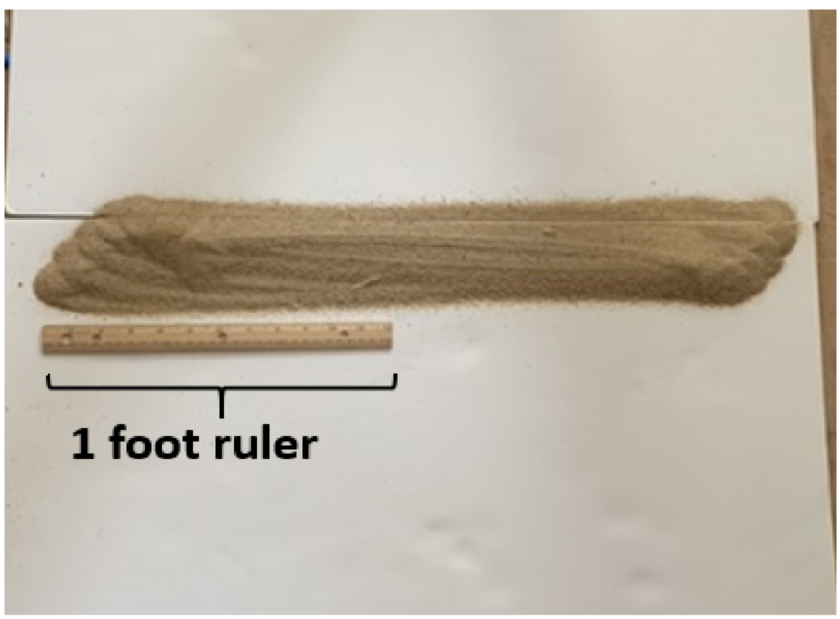

Figure 17: Curve Set 6