NUMERICALLY CALCULATING ORBITS AROUND SCHWARZSCHILD AND KERR BLACK HOLES

by

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ABSTRACT

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When a massive test particle is acted on only by gravity, it follows a timelike geodesic in spacetime. These geodesics are calculated by solving the geodesic equation and are critically dependent on the metric of the spacetime the particle is in. Because of the often complicated nature of the metrics used, numerical prescriptions in codes such as the Java interface *GRorbits.jar* are often used to solve the geodesic equation to find particle orbits. However, *GRorbits.jar* only calculates particle orbits in two dimensional Schwarzschild and Kerr spacetime. In this project, a *Mathematica* code for calculating orbits in three dimensional Schwarzschild spacetime as well as three dimensional Kerr spacetime was developed. Unlike *GRorbits.jar*, this *Mathematica* code also has the ability to calculate orbits in three dimensional Kerr spacetime for particles starting off the equatorial plane. The two dimensional orbits calculated in this project are compared to *GRorbits.jar* by calculating percent differences in the radial and azmuthal coordinates. This analysis shows agreement to *GRorbits.jar* to within tenths of a percent. However, there are a few edge cases where the two codes differ both from each other and the theory. The three dimensional Kerr spacetime portion of the code is compared to an existing Wolfram Demonstrations Project by David Saroff.

ACKNOWLEDGEMENTS

First off I would like to thank my advisor Dr. Gore for his guidance and patience throughout this project. He has been a constant source of advice during the entirety of my undergraduate career. For this, I am very grateful.

I would also like to show my gratitude to all the physics faculty at CNU for bestowing on me the best physics education I could ever ask for. I believe it has prepared me very well for graduate school and all my future endeavors.

Of course thank you to all the friends I made along the way. Through the sleepless nights in the library finishing homework, as well as the even more sleepless nights during "thermo Thursday", they have always been by my side.

Finally, thank you to my mother Judy and my father Iain. Without them I would not be where I am today.

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Contents

Spacetime tells matter how to move; matter tells spacetime how to curve.

—John Archibald Wheeler

1   Introduction

Timelike geodesics are trajectories in spacetime which maximize the proper time, τ, between two points in spacetime. The proper time τ between two points in spacetime is the time measured by a clock traveling on a path through spacetime (referred to as a worldline) which connects those two points. Timelike geodesics are trajectories a massive free particle will follow. A free particle is a particle only acted on by gravitational forces. The equations of motion of a particle on a geodesic can be calculated as a function of its proper time τ by using the geodesic equation. The geodesic equation is a set of four —often coupled—ordinary differential equations. One for each coordinate in the chosen coordinate system plus time. The solutions to the geodesic equation are highly dependent on the chosen metric (i.e what spacetime one is in). The two metrics I will study for this project are the Schwarzschild and Kerr metrics. The Schwarzschild metric describes the spacetime outside of a non-rotating black hole and the Kerr metric describes the spacetime outside of a rotating black hole. It should be noted that the terms geodesic and orbit will be used interchangeably.

It can be shown that orbits in Schwarzschild spacetime will remain in the same plane for all time. This is also the case for orbits initially in the equatorial plane ($\theta = \pi/2$) in Kerr spacetime. However, the Kerr metric is not spherically symmetric. Thus, orbits not initially in the equatorial plane will not stay in the same plane for all time. The purpose of this project was to plot numerical solutions to the geodesic equation found using Mathematica for the metrics describing the spacetime outside of a Schwarzschild and Kerr black hole. Specifically, non-equatorial orbits in Kerr spacetime are illustrated and investigated for any periodic behavior. The plots that this project produced were be compared to [1] and [2] to assess validity.

2 Theory

2.1 The Einstein Equation

As the quote in the Introduction implies, a massive body will curve the spacetime around it. This curvature is calculated using the Einstein equation

$$G^{\mu\nu} = 8\pi G T^{\mu\nu}$$

where $G^{\mu\nu}$ is a tensor describing the curvature of spacetime, G is Newton's gravitational constant, and $T^{\mu\nu}$ is a tensor describing the density and flow of mass and energy at a point in spacetime. The solutions to the Einstein equation yield the metrics that relate the physical separation between two points in spacetime to their coordinate separation. The two metrics that were considered in this project were the metrics which describe Schwarzschild and Kerr spacetime.

2.2 The Geodesic Hypothesis and The Geodesic Equation

The geodesic hypothesis of general relativity states

A free particle follows a geodesic in spacetime.

A free particle is a particle only under the influence of gravitational interactions and a geodesic is a path through spacetime which minimizes a particles proper time between two points in spacetime. In order to calculate geodesics, the geodesic equation must be solved for the components x^μ.

$$\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\mu}{d\tau}\right) - \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\mu}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \tag{1}$$

where $g_{\mu\nu}$ is the metric describing the spacetime that the object is in, and τ is the particle's proper time. Equation (1) is what will be numerically solved to plot orbits in Schwarzschild and Kerr spacetime.

2.3 Schwarzschild Spacetime

The Schwarzschild solution to the Einstein equation is given by the Schwarzschild metric shown below

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta)d\phi^2. \qquad (2)$$

This metric is spherically symmetric, time-independent, and describes the spacetime outside of a spherical, static object such as a Schwarzschild black hole.

2.3.1 Particle Orbits in Schwarzschild Spacetime

A Schwarzschild black hole is a non-rotating, uncharged black hole with radius

$$r_s = 2GM.$$

Using the geodesic equation, one can define two conserved quantities for geodesics around a Schwarzschild black hole. First is the particle's specific energy, e, defined as

$$e \equiv \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau}. \qquad (3)$$

There the particle's total energy is

$$\tilde{E} = \frac{1}{2}(e^2 - 1).$$

Next is the particle's specific angular momentum, l, defined as

$$l \equiv r^2 \sin^2(\theta)\frac{d\phi}{d\tau}. \qquad (4)$$

With these, one can define an effective potential in Schwarzschild spacetime which depends only on the circumferential radial coordinate, r, and l

$$\tilde{V}(r) = -\frac{GM}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{r^3}. \tag{5}$$

The intersection between equation (5) and the particles total energy $\tilde{E}$ give the limits on the radial distance the particle can reach from the black hole. Points where $\tilde{V}'(r) = 0$ represent either stable or unstable circular orbits.

Through explicit computation of the geodesic equation, two more equations of motion can be derived:

$$\frac{d^2r}{d\tau^2} = -\frac{GM}{r^2} + \frac{l^2}{r^3} - \frac{3GMl^2}{r^4} \tag{6}$$

$$0 = r^2\frac{d^2\theta}{d\tau^2} + 2r\frac{dr}{d\tau}\frac{d\theta}{d\tau} - r^2\sin(\theta)\cos(\theta)\left(\frac{d\phi}{d\tau}\right)^2. \tag{7}$$

Now, with equations (3), (4), (6), and (7) as well as the specification of the initial conditions, all geodesics in Schwarzschild spacetime can be numerically calculated.

2.4   Kerr Spacetime

The Kerr solution to the Einstein equation describes the spacetime around a rotating black hole and is shown below in Boyer-Lindquist coordinates

$$ds^2 = -\left(1 - \frac{2GMr}{r^2 + a^2\cos^2(\theta)}\right)dt^2 + \left(\frac{r^2 + a^2\cos^2(\theta)}{r^2 - 2GMr + a^2}\right)dr^2 + (r^2 + a^2\cos^2(\theta))d\theta^2 \tag{8}$$
$$+ \left(r^2 + a^2 + \frac{2GMra^2\sin^2(\theta)}{r^2 + a^2\cos^2(\theta)}\right)\sin^2(\theta)d\phi^2 - \left(\frac{4GMra\sin^2(\theta)}{r^2 + a^2\cos^2(\theta)}\right)d\phi dt.$$

Where a is the black hole's specific angular momentum.

2.4.1 Particle Orbits in Kerr Black Spacetime

Similar to a Schwarzschild black hole, two conserved quantities can be derived using the geodesic equation. The particles specific energy e

$$e \equiv -\left(1 - \frac{2GMr}{r^2 + a^2 \cos^2(\theta)}\right)\frac{dt}{d\tau} - \left(\frac{4GMra\sin^2(\theta)}{r^2 + a^2 \cos^2(\theta)}\right)\frac{d\phi}{d\tau}. \tag{9}$$

As well as the particles specific angular momentum l

$$l \equiv \left(\frac{4GMra\sin^2(\theta)}{r^2 + a^2 \cos^2(\theta)}\right)\frac{dt}{d\tau} + \left(r^2 + a^2 + \frac{2GMra^2\sin^2(\theta)}{r^2 + a^2 \cos^2(\theta)}\right)\sin^2(\theta)\frac{d\phi}{d\tau}. \tag{10}$$

The effective potential in Kerr spacetime depends on both the specific angular momentum as well as the specific energy and is given by

$$\tilde{V}(r) = -\frac{GM}{r} + \frac{l^2 + a^2(1 - e^2)}{2r^2} - \frac{GM(l - ea)^2}{r^3}. \tag{11}$$

Similar to the Schwarzschild case, the intersection between equation (11) and the particles total energy $\tilde{E}$ will give the limits on the radial distance the particle can reach from the black hole and points where $\tilde{V}'(r) = 0$ represent either stable or unstable circular orbits.

Using the geodesic equation, two more equations of motion can be derived:

$$\frac{d}{d\tau}\left(g_{rr}\frac{dr}{d\tau}\right) - \frac{1}{2}\left[\frac{\partial g_{tt}}{\partial r}\left(\frac{dt}{d\tau}\right)^2 + \frac{\partial g_{rr}}{\partial r}\left(\frac{dr}{d\tau}\right)^2\right.$$
$$\left. + \frac{\partial g_{\theta\theta}}{\partial r}\left(\frac{d\theta}{d\tau}\right)^2 + \frac{\partial g_{\phi\phi}}{\partial r}\left(\frac{d\phi}{d\tau}\right)^2 + 2\frac{\partial g_{\phi t}}{\partial r}\frac{d\phi}{d\tau}\frac{dt}{d\tau}\right] = 0 \tag{12}$$

$$\frac{d}{d\tau}\left(g_{\theta\theta}\frac{d\theta}{d\tau}\right) - \frac{1}{2}\left[\frac{\partial g_{tt}}{\partial\theta}\left(\frac{dt}{d\tau}\right)^2 + \frac{\partial g_{rr}}{\partial\theta}\left(\frac{dr}{d\tau}\right)^2\right.$$
$$\left. + \frac{\partial g_{\theta\theta}}{\partial\theta}\left(\frac{d\theta}{d\tau}\right)^2 + \frac{\partial g_{\phi\phi}}{\partial\theta}\left(\frac{d\phi}{d\tau}\right)^2 + 2\frac{\partial g_{\phi t}}{\partial\theta}\frac{d\phi}{d\tau}\frac{dt}{d\tau}\right] = 0. \tag{13}$$

With equations (9), (10), (12), and (13), as well as the appropriate initial conditions, geodesics around a Kerr black hole can be numerically calculated.

2.4.2   The Ergoregion and Event Horizon of a Kerr Black Hole

While a Kerr black hole has a event horizon just like a Schwarzschild black hole, the location as well as the number of event horizons is different. A Kerr black hole has two event horizons denoted r_+ and r_- and are located at

$$r_\pm = GM \pm \sqrt{(GM)^2 - a^2}. \tag{14}$$

However, because the r_- event horizon is located inside the r_+ event horizon, this paper will only consider the event horizon located at r_+.

Another difference in Kerr spacetime is the existence of two infinite-redshift surfaces at

$$r = GM \pm \sqrt{(GM)^2 - a^2 \cos^2(\theta)}. \tag{15}$$

However, the minus root of this equation is inside the event horizon. Therefore, this paper will only consider the positive root.

$$r_e = GM + \sqrt{(GM)^2 - a^2 \cos^2(\theta)} \tag{16}$$

The region of spacetime by this radius is called the ergoregion. Particles located within this region must rotate with the black hole.

2.5   Data Analysis

The Java interface *GRorbits.jar* only computes orbits in two dimensional Schwarzschild and Kerr spacetimes. When an orbit is calculated, this program outputs data for the radius and the azmuthal angle as t in incremented and τ is calculated. The data from *GRorbits.jar*

was downloaded to a text file and imported into Mathematica using the Import command. It was then turned into a function using the InterpolatingFunction command which creates a function out of a list of data through interpolation. The percent difference between the Mathematica orbit and the *GRorbits.jar* orbit will then be calculated and plotted as a function of proper time. Specifically, the percent difference between the r components and the ϕ components will be calculated using the following equations

$$\text{r \% Difference} = \frac{2|r_{GRO} - r_{Mathematica}|}{r_{GRO} + r_{Mathematica}} \times 100 \tag{17}$$

$$\phi \text{ \% Difference} = \frac{2|\phi_{GRO} - \phi_{Mathematica}|}{\phi_{GRO} + \phi_{Mathematica}} \times 100. \tag{18}$$

3   Methods

3.1   Numerically Calculating Orbits in Schwarzschild Spacetime

Using equations (3), (4), (6), and (7), as well as the appropriate initial conditions, all orbits in Schwarzschild spacetime can be calculated. Also, with equation (5), the effective potential of orbits in the equatorial plane can be calculated and plotted. However, equations (3), (4), (6), and (7) are coupled differential equations. Therefore, *Mathematica's* NDSolve command was used in order to numerically derive the geodesics in Schwarzschild spacetime. The NDSolve command in *Mathematica* has the syntax

$$NDSolve[\{eqn_1, eqn_2, ...\}, \{u_1, u_2, ...\}, \{t, t_{min}, t_{max}\}].$$

This command outputs a list of interpolating functions which are solutions to the geodesic equation found using interpolation. These solutions are then fed into a ParametricPlot3D command which is nested inside a Manipulate command in order for the user to progress the proper time of the particle as they wish.

In the output for this program is also a drawing of the Schwarzschild black hole itself.

This is accomplished through using *Mathematica's* ParametricPlot3D command as well as the equation for the radius of a Schwarzschild black hole ($r_s = 2GM$) to create a black sphere at the origin of the coordinate system. The plots of the solutions for the orbit as well as the black hole are all shown on a single three dimensional plot using the Show command.

Finally, the effective potential was graphed using the Plot command and the intersections with the total energy were found using the NSolve command. Also, the points which represent the circular orbits were found solving the differential equation $\tilde{V}'(r) = 0$ using the NDSolve command.

3.2 Numerically Calculating Orbits in Kerr Spacetime

In much a similar fashion to Schwarzschild spacetime, orbits in Kerr spacetime are calculated using equations (9), (10), (12), and (13) as well as the appropriate initial conditions. Also, with equation (11), the effective potential of orbits in the equatorial plane can be calculated and plotted. *Mathematica's* NDSolve command was used in order to numerically derive the geodesics in Kerr spacetime.

The solutions in Kerr spacetime, which are interpolating functions, are solutions to the geodesic equation found using interpolation. These solutions are then fed into a ParametricPlot3D command which is nested inside a Manipulate command in order for the user to progress the proper time of the particle as they wish.

The drawing of the Kerr black hole is accomplished in the same way as the drawing of the Schwarzschild black hole. Except, in this case, the positive root of equation (14) is used to calculate the radius of the black hole. Also, the ergoregion is represented as a gray region surrounding the black hole and has the radius given in equation (16). The ergoregion is plotted using *Mathematica's* SphericalPlot3D command. The plots of the solutions for the orbit as well as the black hole are all shown on a single three dimensional plot using the Show command.

Finally, the effective potential was graphed by simply using the Plot command and the

intersections with the total energy were found using the NSolve command. Also, the points which represent the circular orbits were found solving the differential equation $\tilde{V}'(r) = 0$ using the NDSolve command.

3.3   Moving Data from GRorbits.jar into Mathematica

In order to make the comparison to *GRorbits.jar*, the data for an orbit in *GRorbits.jar* must be exported from the program into *Mathematica*. To do this, a Python program was created to format the data from *GRorbits.jar* into a form *Mathematica* would accept. The data was copied and pasted into a text file, the data for the radius, azmuthal angle, and proper time were extracted using the Python program, then dumped into a text file to then import into *Mathematica*. For each orbit that was to be compared this was done. The data was then fed into *Mathematica* using the Import command. this command takes an external data file and stores it into the *Mathematica* program. Next, the Interpolation function was used on the data to create $r_{GRO}(\tau)$ which is the radius as a function of the proper time as well as $\phi_{GRO}(\tau)$ which is the azmuthal angle as a function of the proper time. These functions were then plotted around black holes to create similar plots as in sections 3.1 and 3.2. Finally, equations (17) and (18) were used to plot the percent difference as a function of proper time for each orbit that was analyzed.

3.4   Comparison to David Saroff's Wolfram Demonstrations Project

The comparison to David Saroff's Wolfram Demonstrations Project proved more difficult. Because of the nature of [1] there is no data to compare, and I could not get my hands on his code. So the comparison to his code was a "eyeball" comparison. Using the same initial conditions in both programs, pictures of the orbits are set side by side and looked at for any striking differences.

4   Data

When comparing the *Mathematica* code to *GRorbits.jar*, six trials were run in each space-time. When comparing the *Mathematica* code to David Saroff's Wolfram Demonstrations Project, four trials were run in three dimensional Kerr spacetime. Each section that follows will contain screenshots of each trial. One section for the trials comparing my *Mathematica* code and *GRorbits.jar* in two dimensional Schwarzschild spacetime. One section for the trials comparing my *Mathematica* code and *GRorbits.jar* in two dimensional Kerr spacetime. Finally, one section for the trials comparing my *Mathematica* code to David Saroff's Wolfram Demonstrations Project in three dimensional Kerr spacetime. In each trial comparing to *GRorbits.jar*, there are pictures of the effective potential for the orbit, the orbit generated by *Mathematica*, as well as plots of the r percent difference and the ϕ percent differences to *GRorbits.jar*. In each trial comparing *Mathematica* to David Saroff's Wolfram Demonstrations Project, the *Mathematica* orbit will be on the left. The initial conditions for each orbit are described in the graph of the effective potential for the two dimensional trials and in the figure captions for the comparison to David Saroff. Throughout all trials it is assumed that $\phi(0) = \phi'(0) = 0$, $r'(0) = 0$, and $\theta'(0) = 0$ unless explicitly stated.

4.1   Mathematica vs. GRorbits.jar: 2D Schwarzschild Orbits
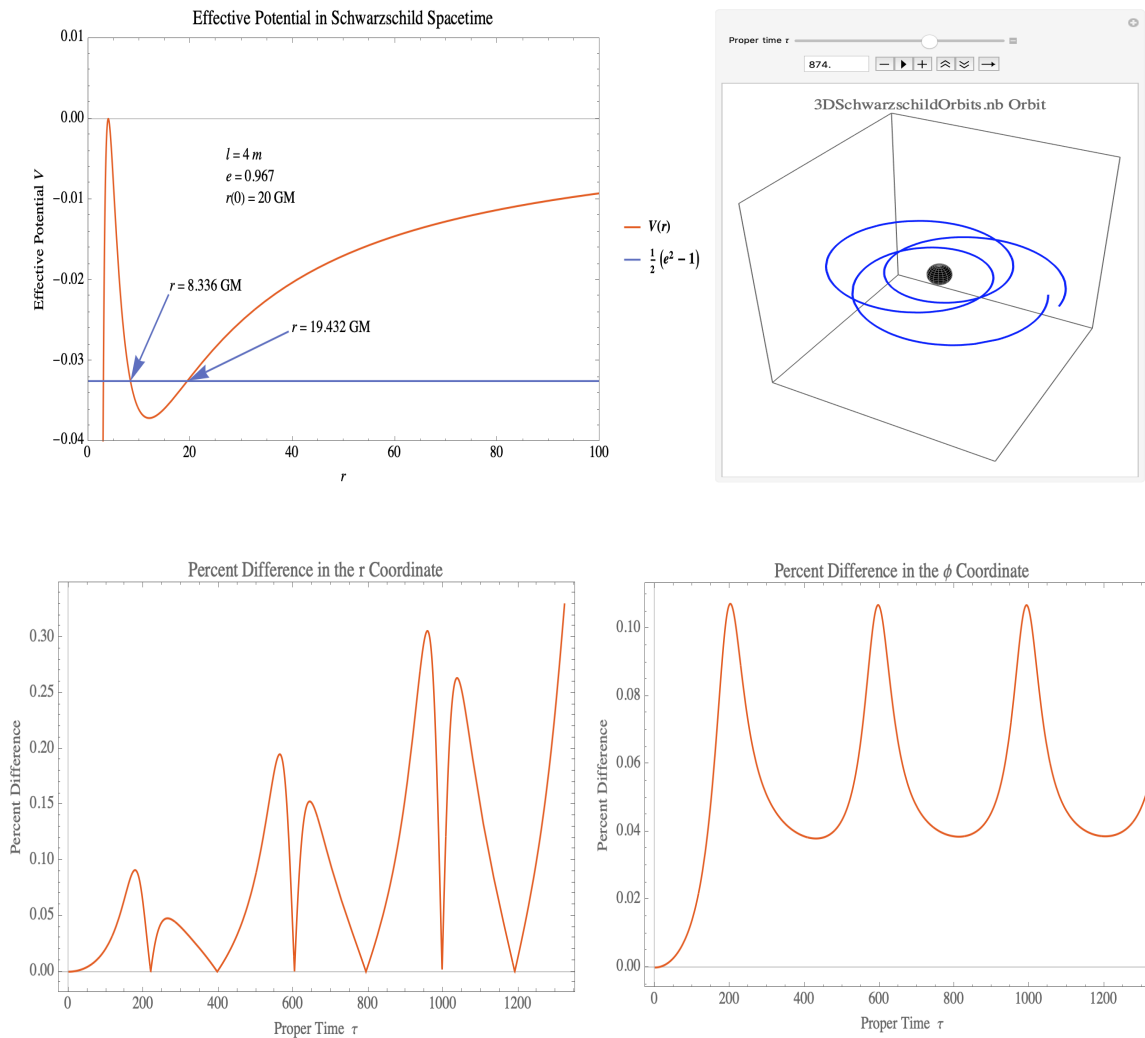
Figure 1: Trial 1 in 2D Schwarzschild spacetime.
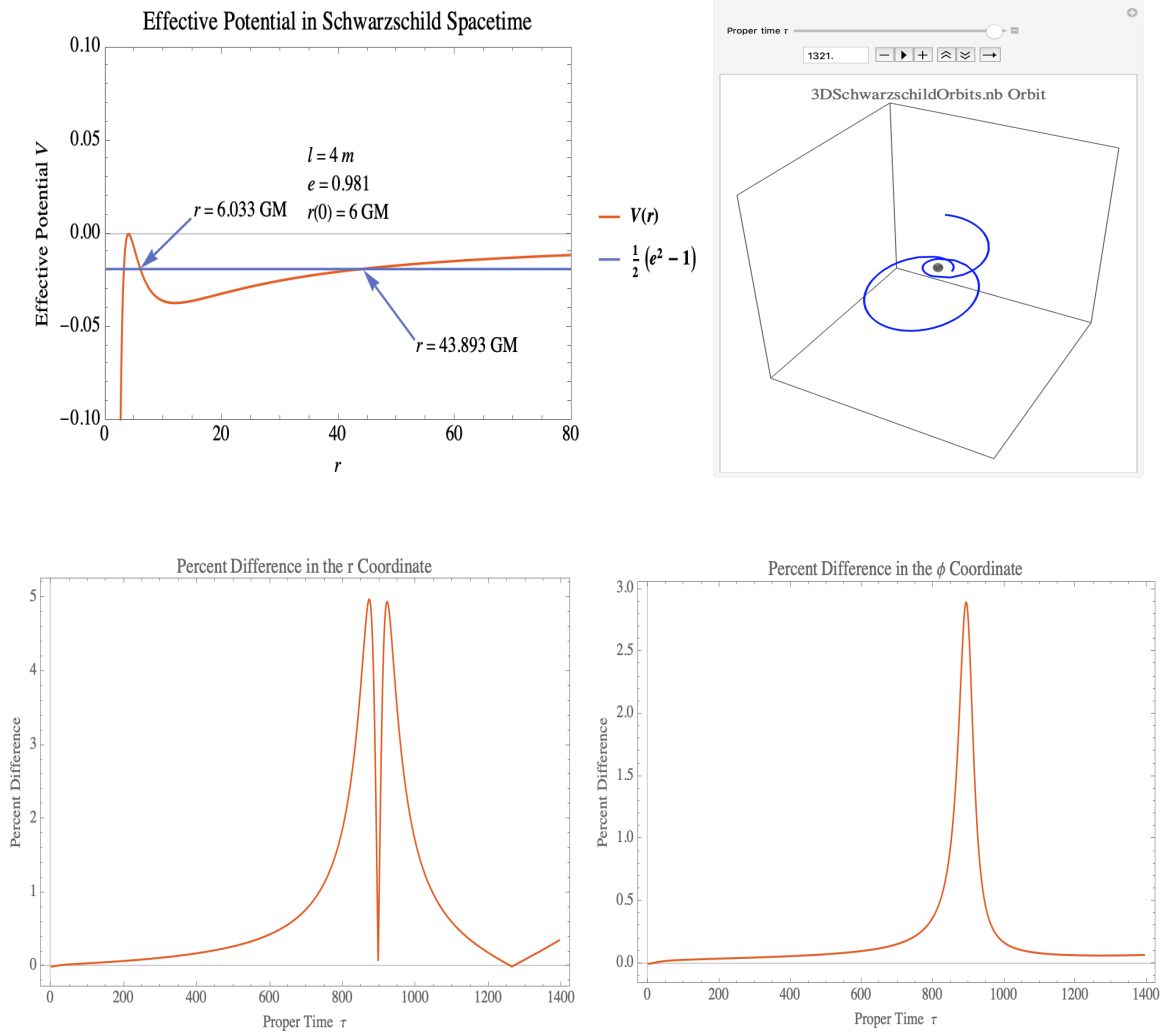
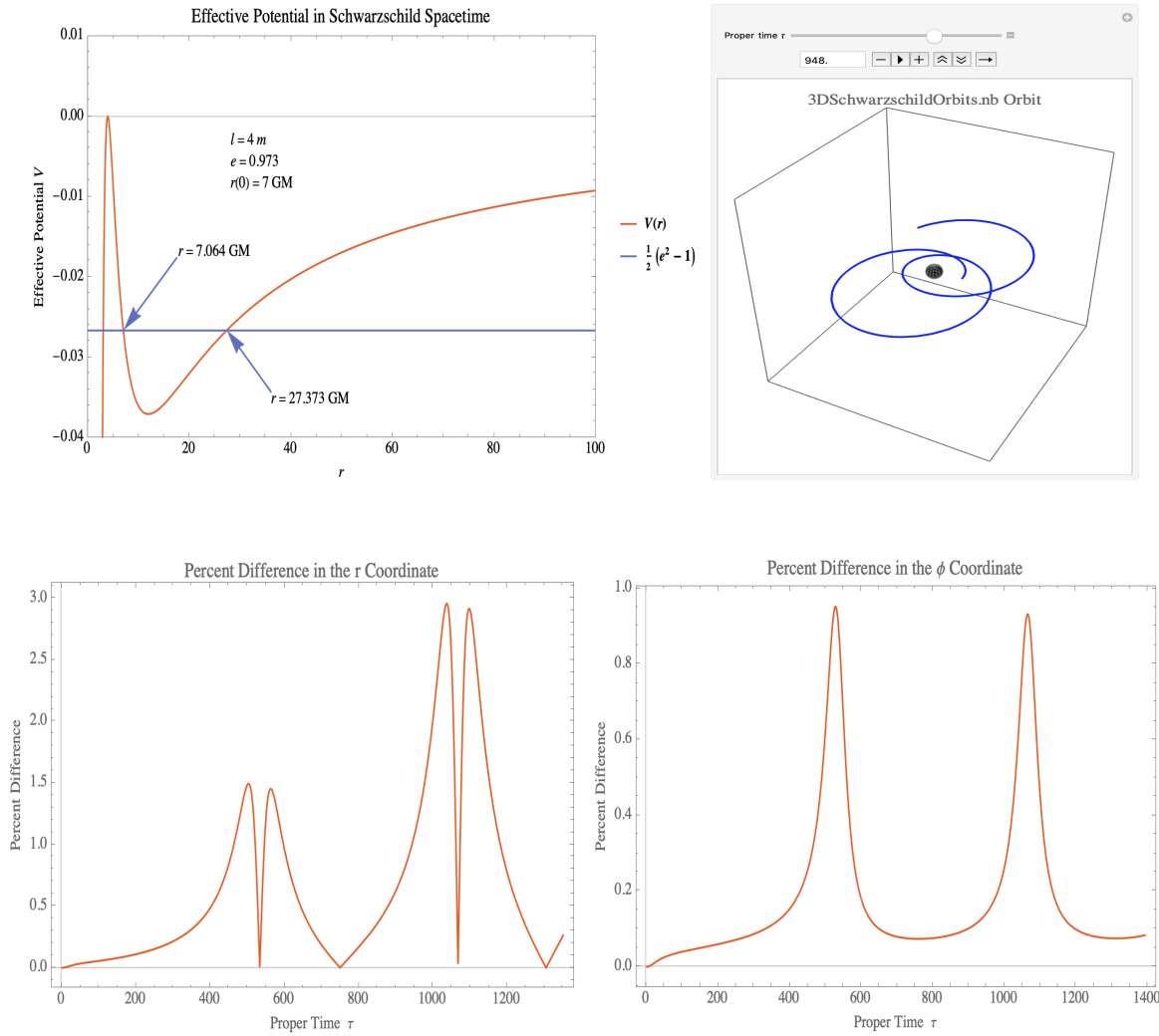Figure 2: Trial 2 in 2D Schwarzschild spacetime.
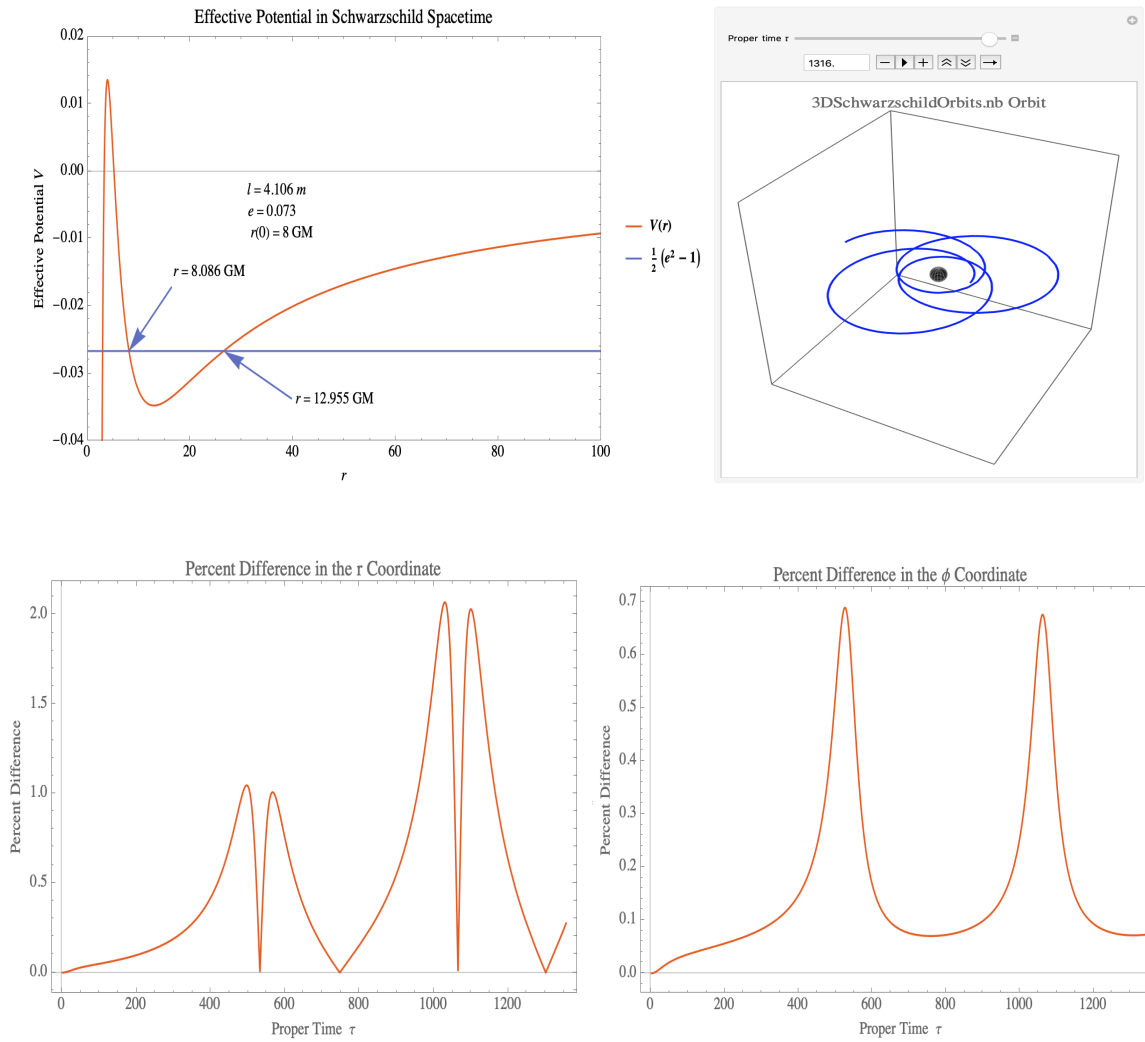
Figure 3: Trial 3 in 2D Schwarzschild spacetime.
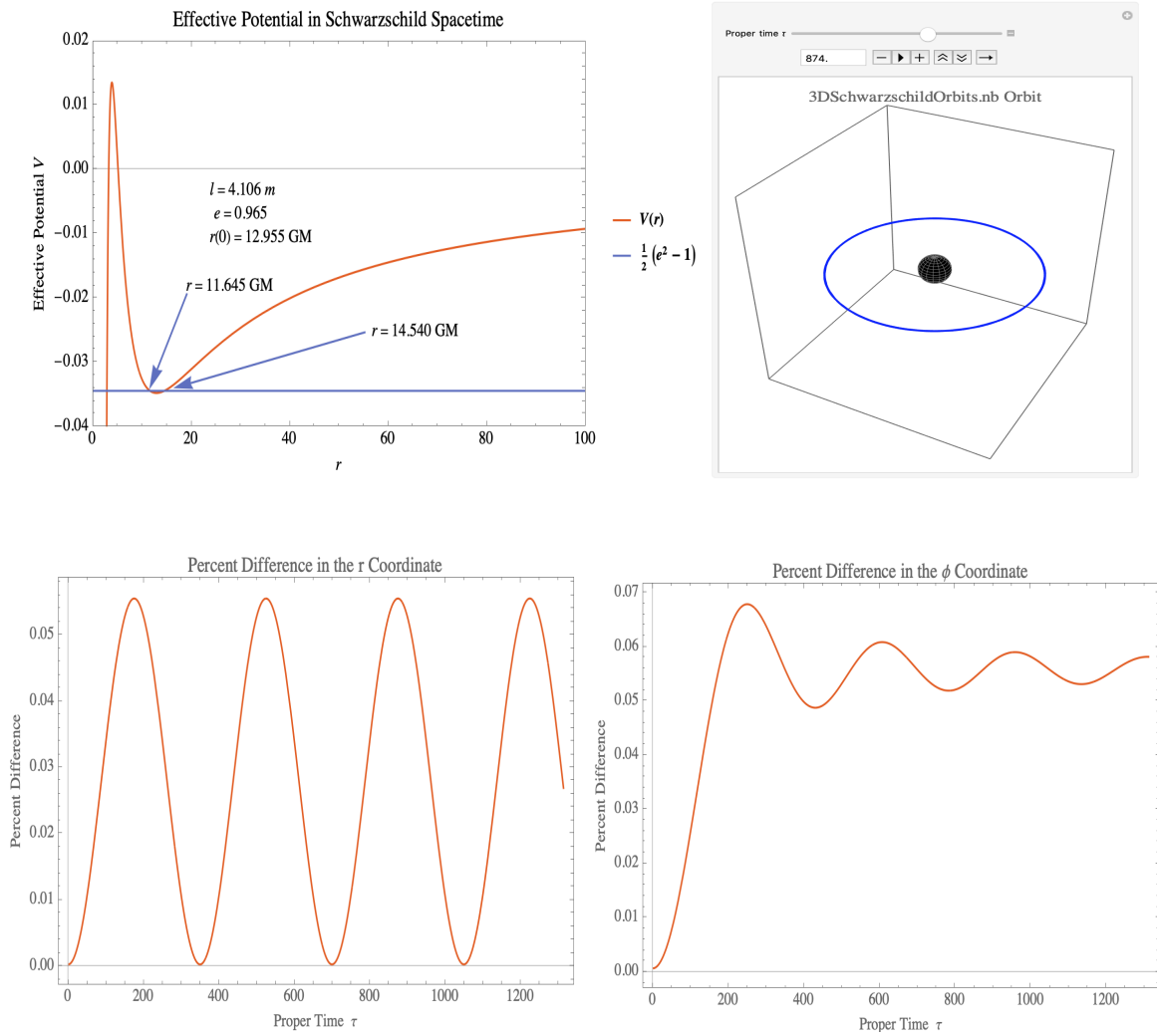
Figure 4: Trial 4 in 2D Schwarzschild spacetime.
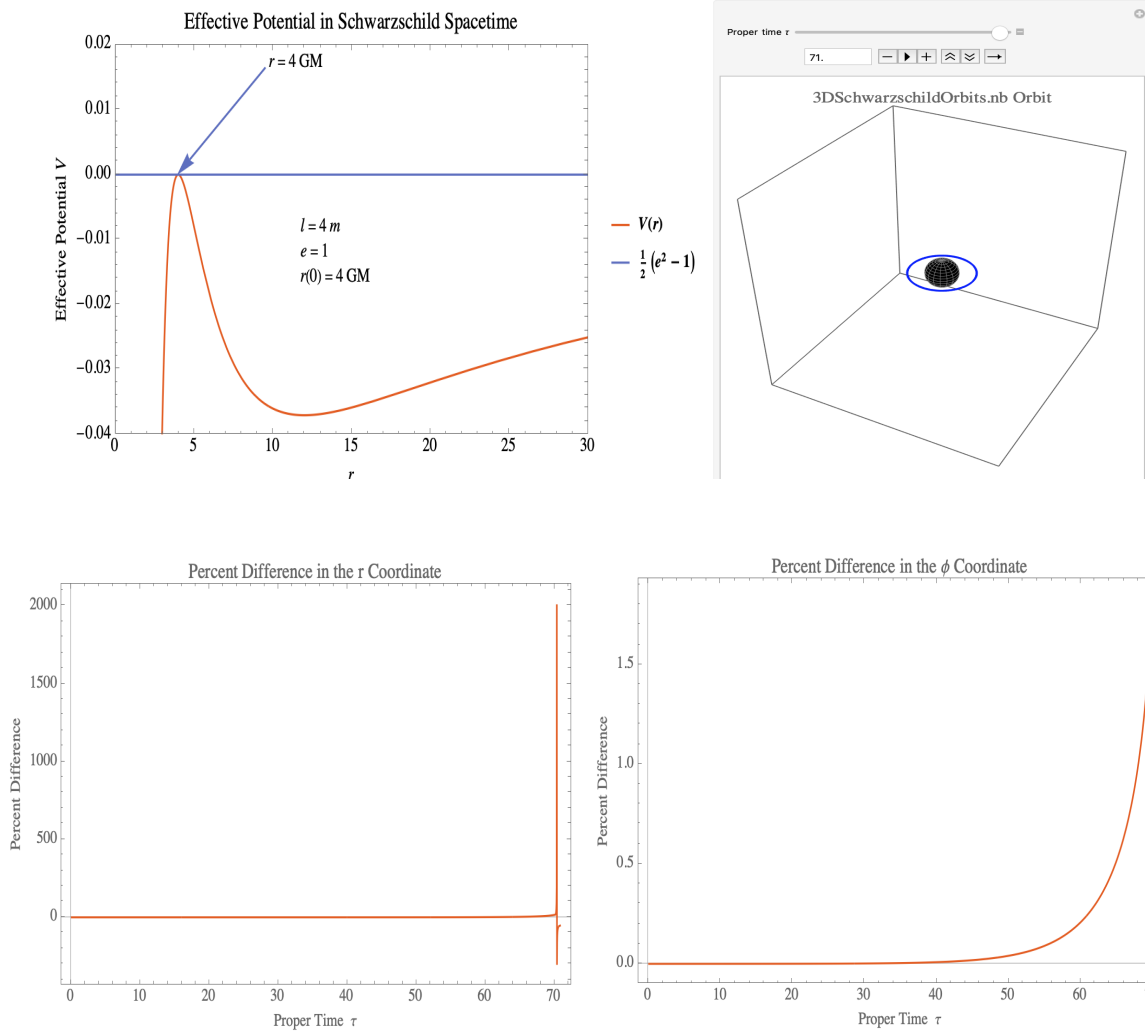
Figure 5: Trial 5 in 2D Schwarzschild spacetime.

Figure 6: Trial 6 in 2D Schwarzschild spacetime.

4.1.1   Discussion: 2D Schwarzschild Spacetime Trials

Most of the trials in 2D Schwarzschild spacetime show good agreement between the *Mathematica* code and *GRorbits.jar*. The percent difference between the r equations never rising above 5% except in trial 6. It is interesting to note the patterns in the percent difference equations. The minima of the r percent differences seem to occur at around the same proper time as the maxima of the ϕ percent differences. Also, the only case where the percent difference blows up is trial 6 which is an edge case. Trial 6 represents an unstable circular orbit. The *Mathematica* code kept the orbit circular, however, *GRorbits.jar* predicted the particle would fall in after around 70 seconds of proper time. Disagreeing with both theory and the *Mathematica* code.

4.2   Mathematica vs. GRorbits.jar: 2D Kerr Orbits
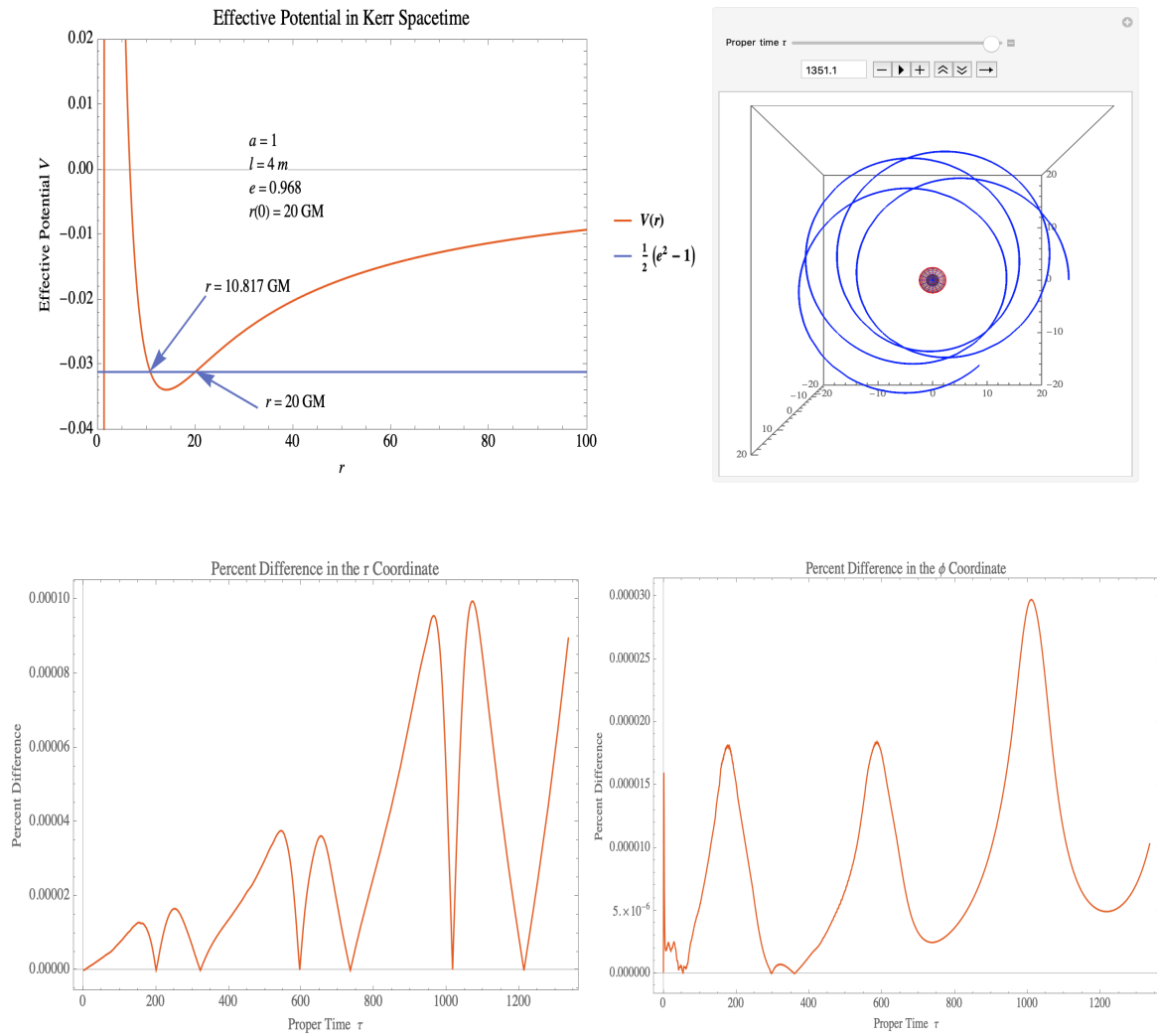
Figure 7: Trial 1 in 2D Kerr spacetime
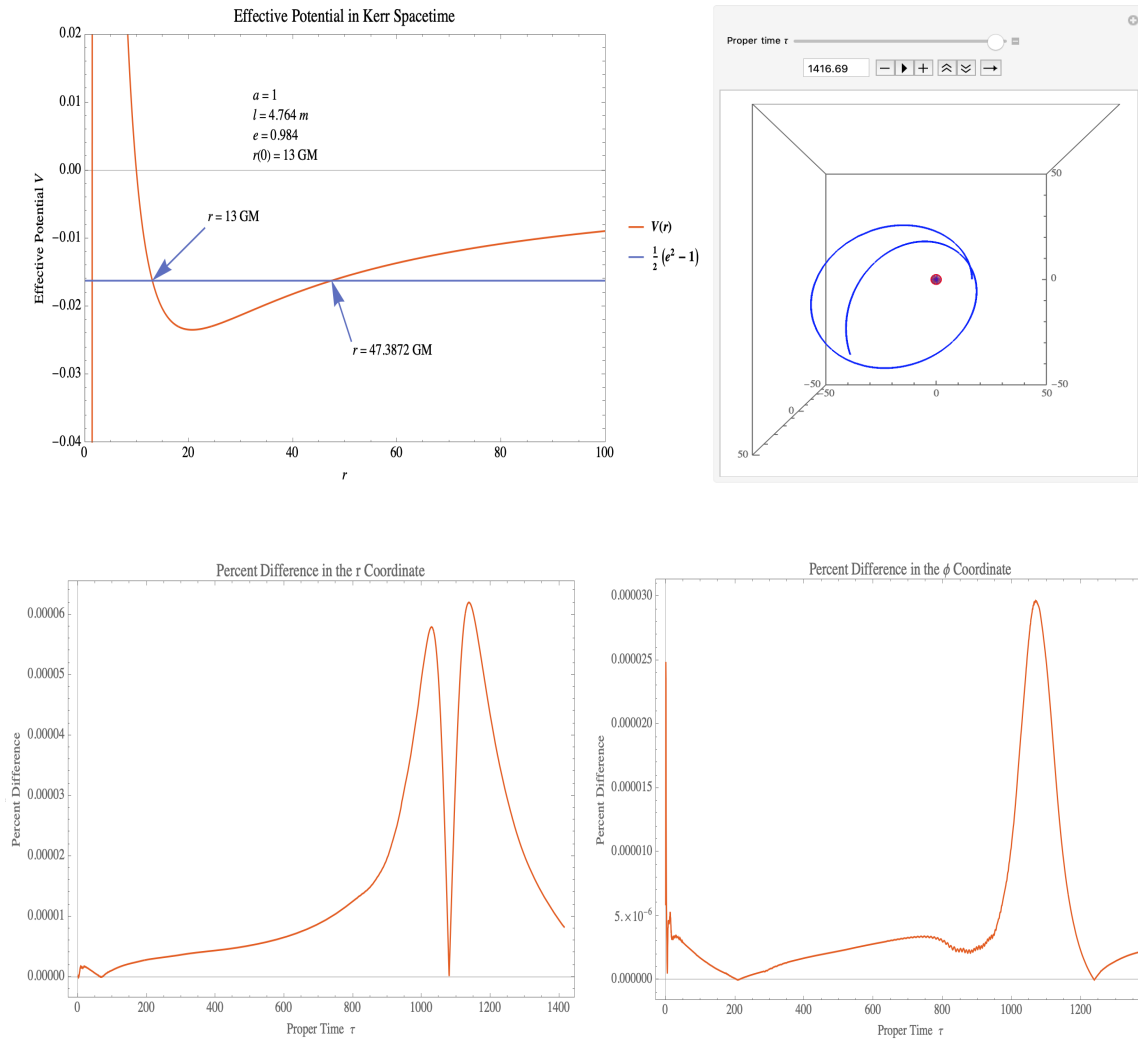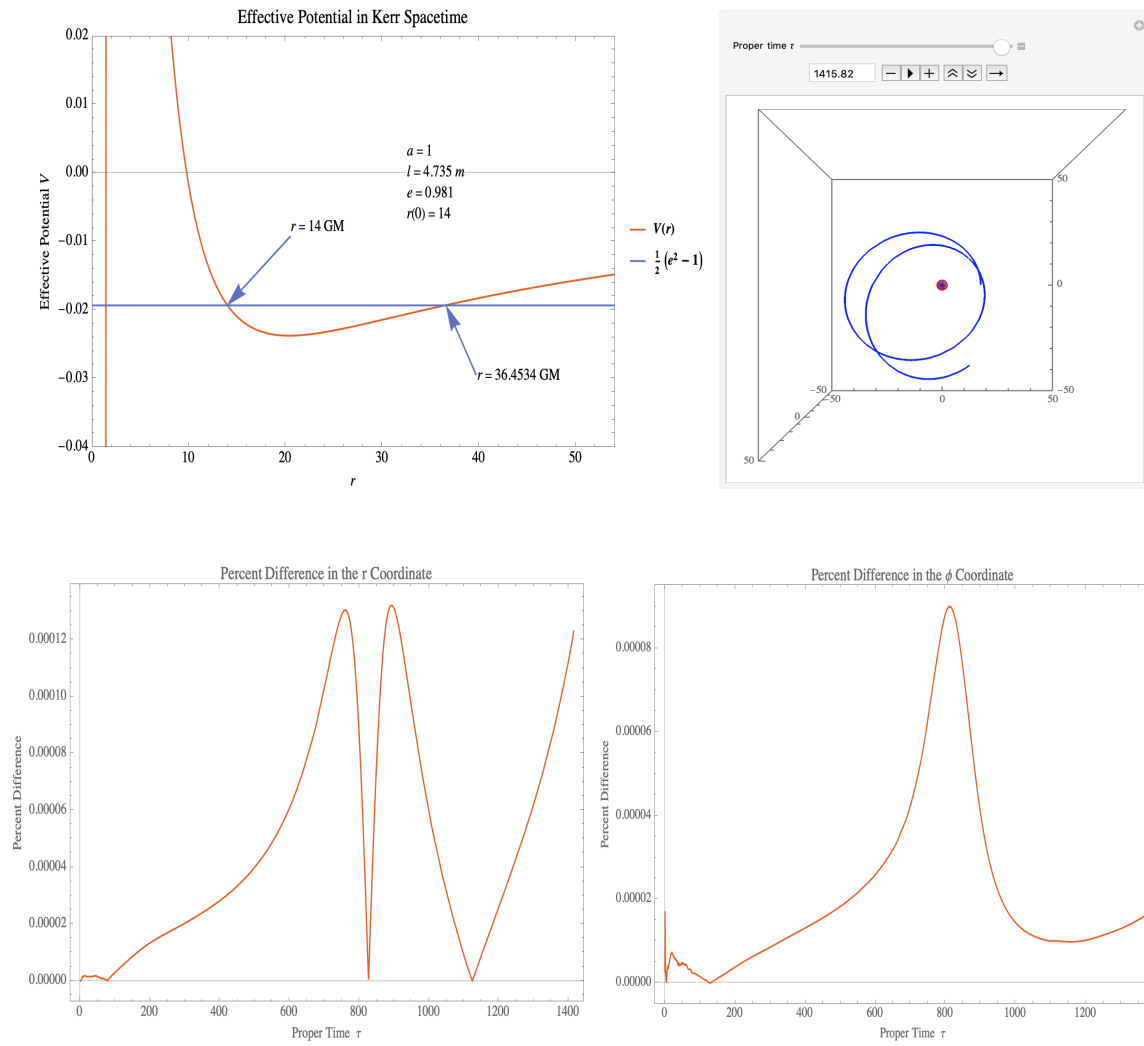
Figure 8: Trial 2 in 2D Kerr spacetime

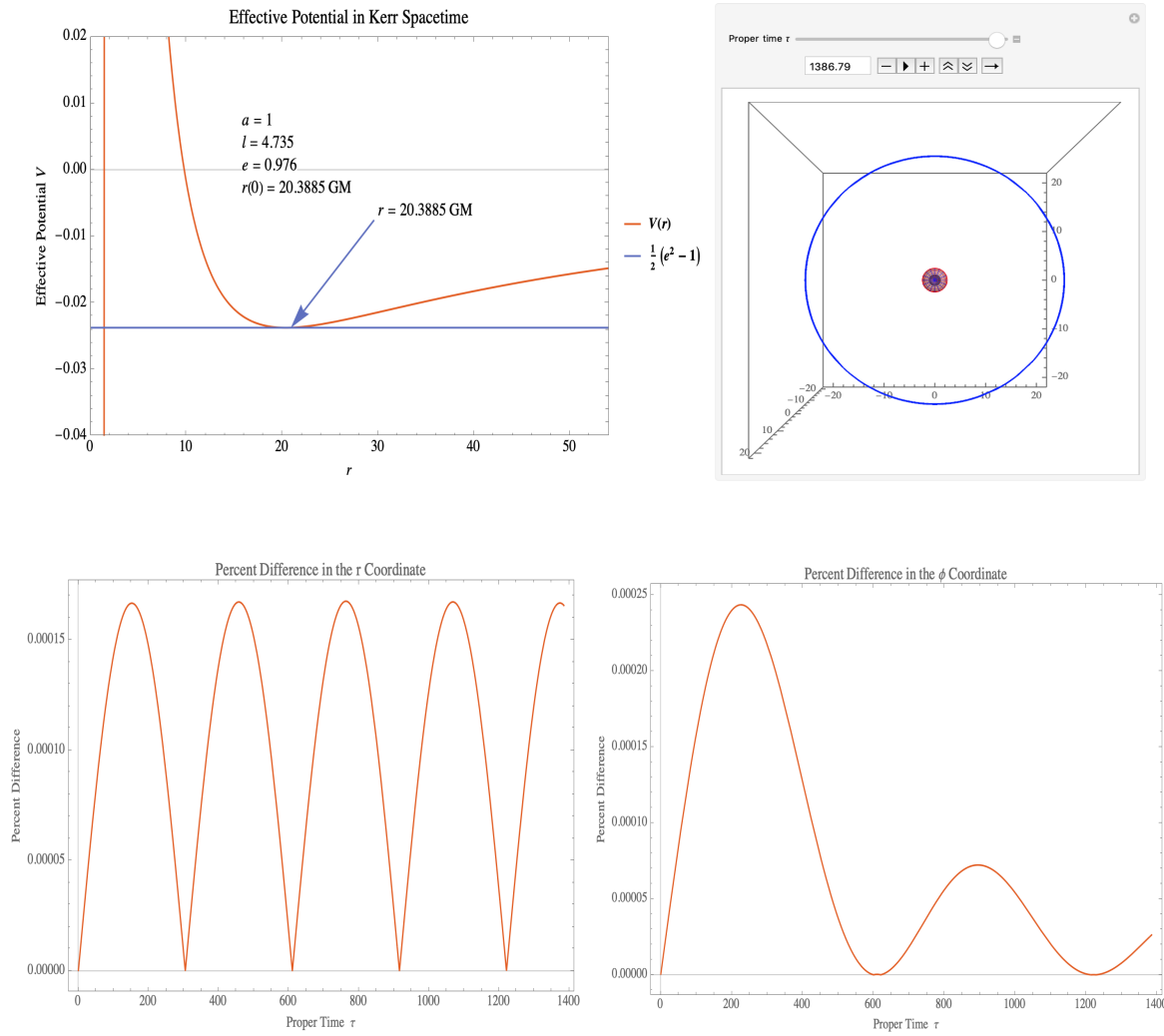Figure 9: Trial 3 in 2D Kerr spacetime

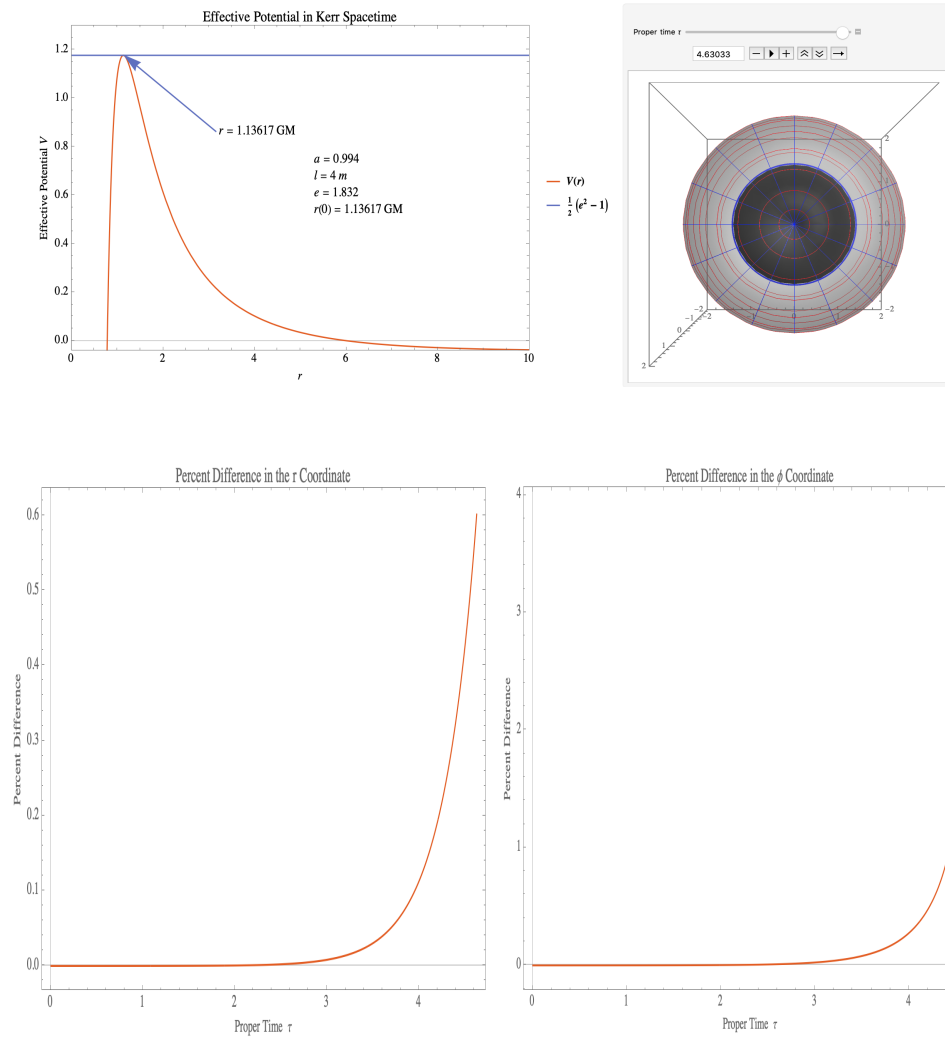Figure 10: Trial 4 in 2D Kerr spacetime
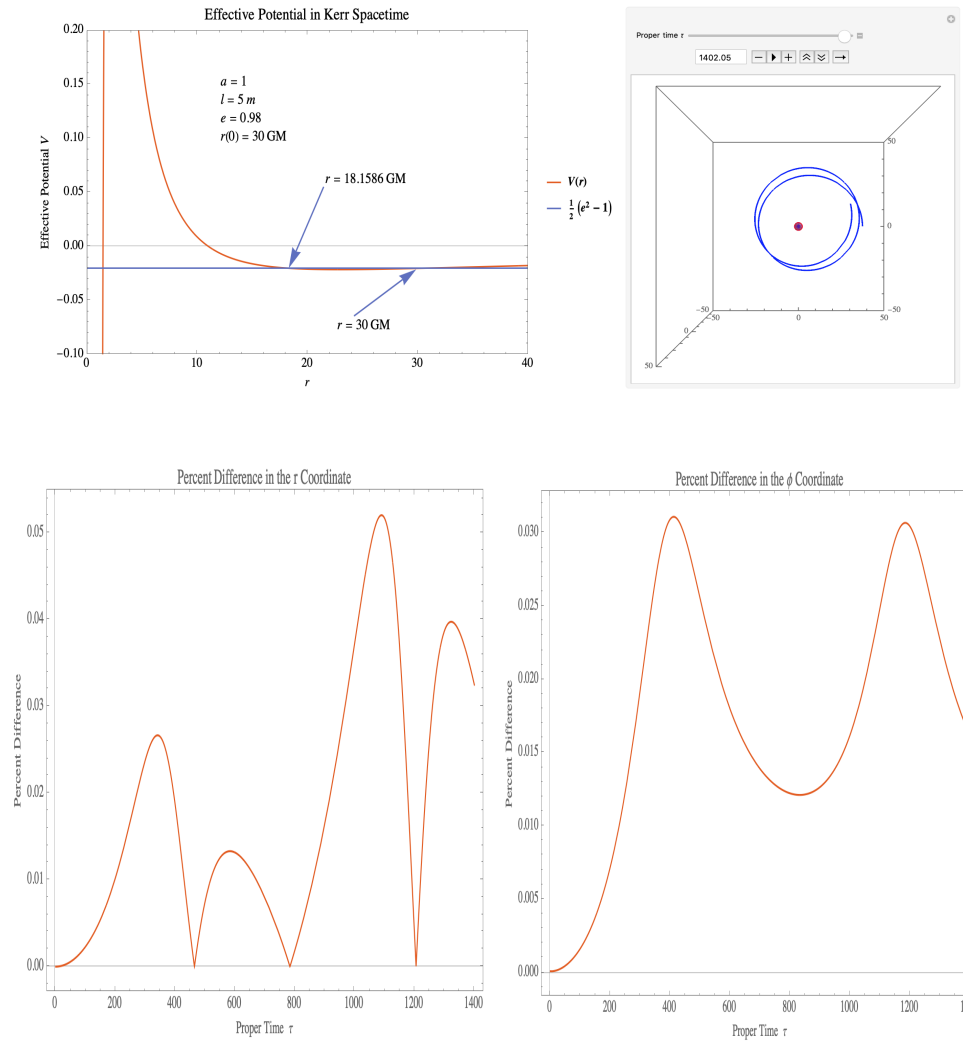
Figure 11: Trial 5 in 2D Kerr spacetime

Figure 12: Trial 6 in 2D Kerr spacetime

4.2.1   Discussion: 2D Kerr Spacetime Trials

The trials run in Kerr spacetime seem to agree better with *GRorbits.jar* than in Schwarzschild spacetime with the r and ϕ percent differences staying well below 1% except in one case. The same pattern from the Schwarzschild trials still holds in that the minima of the r percent difference graphs come around the same proper time as the maxima of the ϕ percent difference graphs. Also, the only large percent error come with the unstable circular orbit in trial 5 where *GRorbits.jar* again deviated from theory and the *Mathematica* code and predicted the particle would fall into the black hole when a circular orbit should have persisted.

4.3   Mathematica vs. David Saroff: Non-Equatorial Kerr Orbits
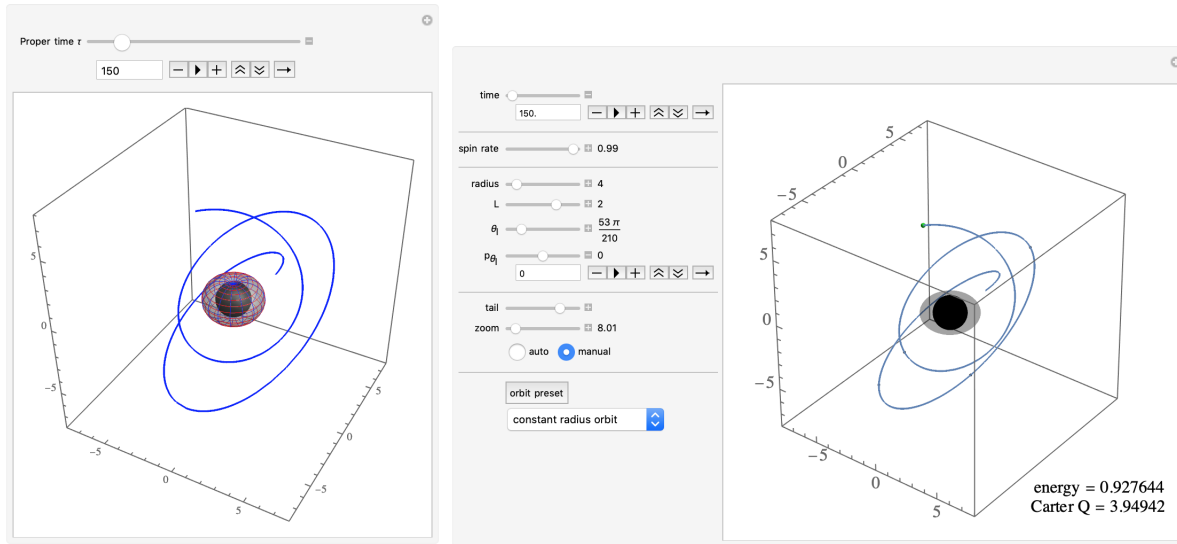

Figure 13: Trial 1 vs David Saroff: $r(0) = 4GM, \theta(0) = 53\pi/210, l = 2m, a = 0.99M, e = 0.927644$.
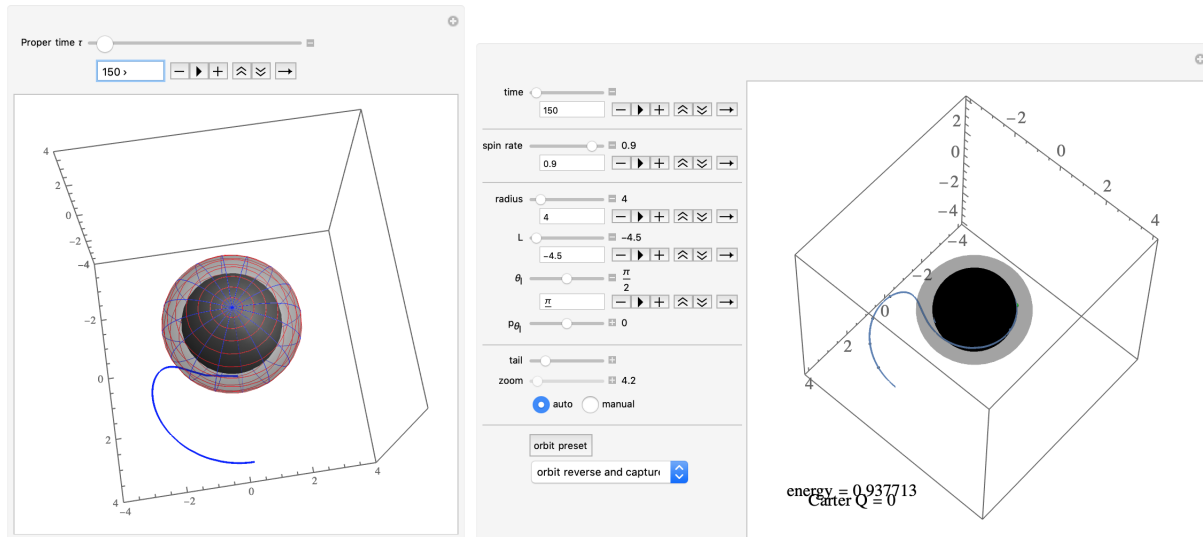
Figure 14: Trial 2 vs David Saroff: $r(0) = 4GM, \theta(0) = \pi/2, l = -4.5\mathrm{m}, a = 0.99M, e = 0.937713$.
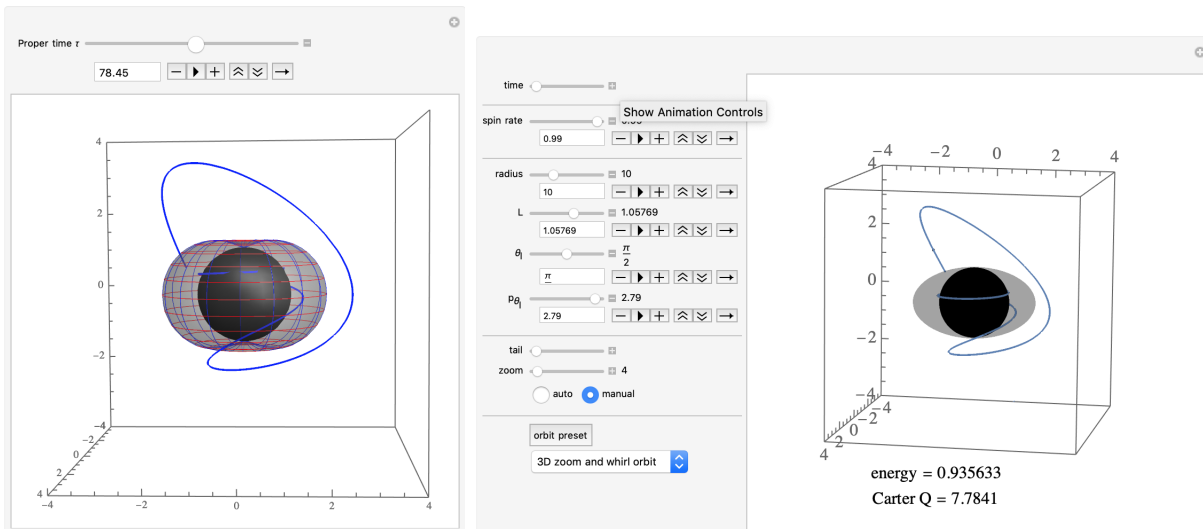

Figure 15: Trial 3 vs David Saroff: $r(0) = 10GM, \theta(0) = \pi/2, \theta'(0) = 0.78, l = 1.05769\mathrm{m}, a = 0.99M, e = 0.935633$.
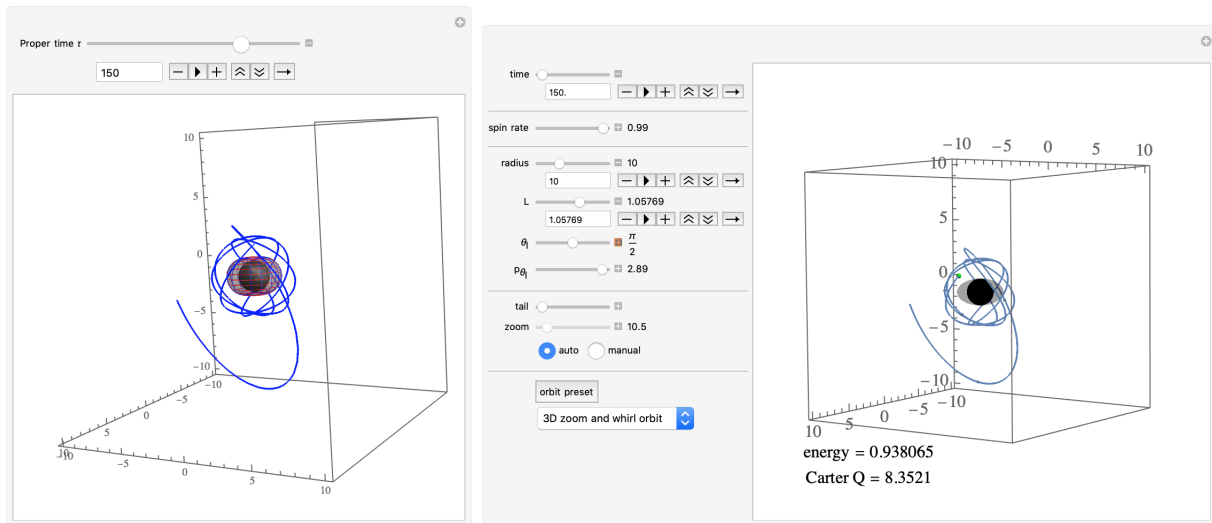
Figure 16: Trial 4 vs David Saroff: $r(0) = 10GM, \theta(0) = \pi/2, l = 1.05769\text{m}, a = 0.99M, e = 0.935633$.

4.3.1 Discussion: Non-equatorial Kerr Orbit Trials

Because of the nature of David Saroff's Wolfram Demonstrations project, no hard comparison was able to be made between the two projects. However, the *Mathematica* code is able to reproduce the results from this project to the naked eye. This is a good sign that the *Mathematica* code is correctly plotting non-equatorial orbits in Kerr spacetime.

5 Orbital periods in Kerr Spacetime

A quick investigation into the orbital period in not equatorial Kerr orbits was done. The nature of one orbital period needed to be defined. For this purpose, one orbital period was completed when the particle returned to its original azimuthal coordinate. That is, multiples of 2π of it's original azimuthal coordinate. Below is a plot of $\phi(\tau)$ and its intersections with even multiples of π for an orbit with initial conditions $r(0) = 13$ GM, $l = 4$ m, $e = 0.978, a = 0.99$, and $\theta(0) = \pi/3$:
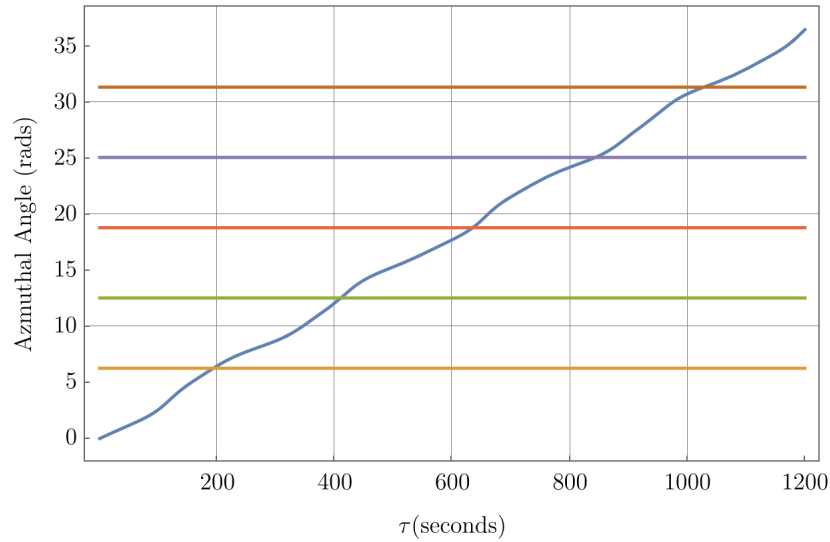
These intersections represent the period of the orbit for that particular cycle. These values can be solved for numerically, yielding

$$\tau_{1,2,\dots,5} = \{193.279, 410.499, 634.754, 843.421, 1028.02\}.$$

The difference in these orbital periods remain in about the range of ~ 200 seconds.

6 Conclusion

The theory behind calculating orbits in two dimensional Schwarzschild spacetime as well as two and three dimensional Kerr spacetime has been presented and the *Mathematica* code used to simulate multiple orbits has been shown to be in good agreement with existing codes. In fact, the results from this simulation agree better with theory than the Java interface *GRorbits.jar* for multiple edge cases in both Schwarzschild and Kerr spacetime. The *Mathematica* code developed here also seems to show very good agreement to an existing code written by David Saroff when calculating orbits in three dimensional Kerr spacetime off the equatorial plane.

References

[1] David Saroff. Wolfram demonstrations project.

[2] tuleja et al. How to run grorbits.