Constructing an Inverted Pendulum

Scott Reid, Advisor: William Phelps

March 13, 2022

1 Abstract

An inverted pendulum is a standard pendulum that is aims to stabilize vertically upward. In the case of the standard pendulum, gravity and friction work together to ultimately bring the pendulum into rest vertically downward, to stabilize vertically without any external forces would defy the laws of physics. This project creates a stable vertical position by applying external forces to the pivot point. The pendulum is attached to a cart, which moves based on the angle the pendulum makes with the vertical relative to the desired vertical position. In order for the cart to know how much to move, the angle data is analyzed by a PID controller, which then sends the commands to drive the cart. As a result of these steps, this project is able to stabilize a pendulum in an inverted position for any given period of time.

2 Introduction

The Newtonian formulation is a vector formulation that is easy to use in an inertial frame of reference with constant forces applied. Once a system becomes non-inertial or undergoes forces that change in magnitude and/or direction, the system becomes extremely difficult to model using Newtonian mechanics. This is why the Lagrangian formulation is important to physics. It uses energy, which is a scalar quantity, to describe a system [1]. This allows for the Lagrangian to eliminate constraint forces that the Newtonian formulation must include [1]. In the case of the inverted pendulum, which will constantly undergo changing forces, the Lagrangian is the best way to model the system.

This project will create an inverted pendulum that is stabilized by applying external forces to the pivot point. This project is significant because it will be an excellent physical manifestation of a system that is best modeled by the Lagrangian formulation, as opposed to the Newtonian formulation. Future PHYS 401 students will be able to solve the equations of motion for an inverted pendulum via the Lagrangian formulation, and then watch those equations play out with this design.

The external force is applied to the pivot point via the stepper motor. The pendulum is attached to a cart that is driven by this stepper motor, which knows precisely how much to move based on the data from the rotary encoder. The rotary encoder measures the angular displacement of the pendulum. This data is read and analyzed by the PID controller, which then tells the stepper motor how to move. I chose these techniques based on research I did into how other groups designed inverted pendulums.https://www.overleaf.com/project/61e89dcfe4128de178e37901

3 Theory

Mathematical Theory

The Newtonian formulation is a vector formulation that is easy to use in an inertial frame of reference with constant forces applied. Once a system becomes non-inertial or undergoes forces that change in magnitude and/or direction, the system becomes extremely difficult to model using Newtonian mechanics. This is why the Lagrangian formulation is important to physics. It uses energy, which is a scalar quantity, to describe a system. This allows for the Lagrangian to eliminate constraint forces that the Newtonian formulation must include. In the case of the inverted pendulum, which will constantly undergo changing forces, the Lagrangian is the best way to model the system.

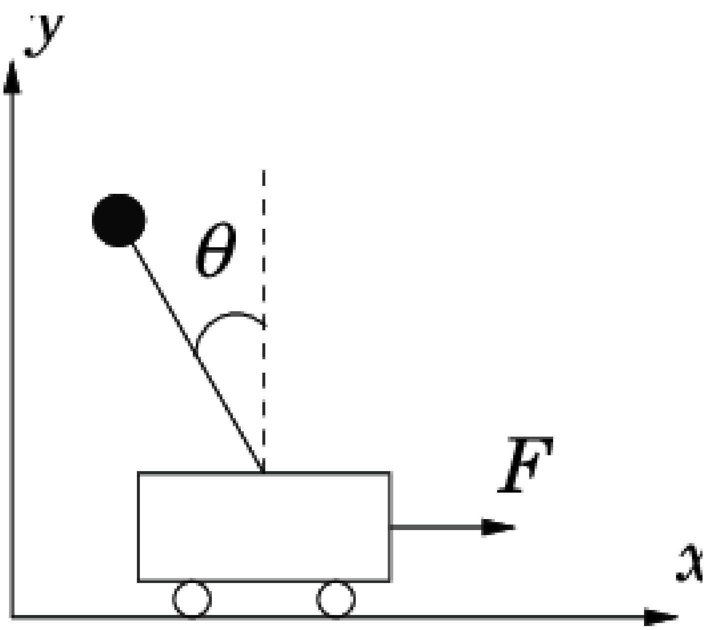

Figure 1: Inverted Pendulum on a Cart Diagram via https://www.researchgate.net

Above is a diagram of an inverted pendulum on a cart, where:

Mass of the cart = M
Mass of the bob + mass of the pendulum = m
Length of the rod = L
Displacement of the cart in the x-direction = X
Displacement of the bob in the x-direction = x
Displacement of the bob in the y-direction = y

The formulas for the positions of the bob are:

$$x = X - Lsin\theta$$
$$y = Lcos\theta$$

We can now find the velocities by taking the time derivatives of the positions:

$$\dot{x} = \dot{X} - Lcos\theta\dot{\theta}$$
$$\dot{y} = -Lsin\theta\dot{\theta}$$

The kinetic energy $T = \frac{1}{2}mv^2$ of the system will be:

$$T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$
$$T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m((\dot{X} - Lcos\theta\dot{\theta})^2 + (-Lsin\theta\dot{\theta})^2)$$
$$T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{X}^2 - 2L\dot{\theta}\dot{X}cos\theta + L^2\dot{\theta}^2(sin\theta^2 + cos\theta^2))$$
$$T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{X}^2 - 2L\dot{\theta}\dot{X}cos\theta + L^2\dot{\theta}^2)$$
$$T = \frac{1}{2}(M+m)\dot{X}^2 + \frac{1}{2}mL^2\dot{\theta}^2 - mL\dot{\theta}\dot{x}cos\theta$$

The potential energy $U = mgh$ will be:

$$U = mgLcos\theta$$

The Lagrangian $\mathcal{L} = T - U$ [1] will then be:

$$\mathcal{L} = \frac{1}{2}(M+m)\dot{X}^2 + \frac{1}{2}mL^2\dot{\theta}^2 - mL\dot{\theta}\dot{X}cos\theta - mgLcos\theta$$

Lagrange's equation when there is a net external force acting on the system [1] is:

$$\frac{d}{dt}\left[\frac{\partial\mathcal{L}}{\partial\dot{q}}\right] - \frac{\partial\mathcal{L}}{\partial q} = F_{net}$$

We now have a formula with two degrees of freedom: position and angle. We can hence solve the Lagrangian for both of these, we will start with θ. Since the net external force will only act on the cart, we can say that for the angle θ, $F_{net} = 0$. Therefore:

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = mL^2\dot{\theta} - mL\dot{X}cos\theta$$
$$\frac{d}{dt}\left[\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right] = mL^2\ddot{\theta} - mL\ddot{X}cos\theta + mL\dot{X}sin\theta\dot{\theta}$$
$$\frac{\partial\mathcal{L}}{\partial\theta} = mgLsin\theta + mL\dot{X}sin\theta\dot{\theta}$$
$$\frac{d}{dt}\left[\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right] - \frac{\partial\mathcal{L}}{\partial\theta} = mL^2\ddot{\theta} - mL\ddot{X}cos\theta - mgLsin\theta = 0$$
$$0 = L\ddot{\theta} - \ddot{X}cos\theta - gsin\theta$$
$$\ddot{\theta} = \frac{\ddot{X}cos\theta + gsin\theta}{L}$$

This leaves us with our first equation of motion:

$$\ddot{\theta} = \frac{\ddot{X}cos\theta + gsin\theta}{L} \tag{1}$$

Now we will differentiate with respect to X:

$$\frac{\partial\mathcal{L}}{\partial\dot{X}} = (M+m)\dot{X} - mL\dot{\theta}cos\theta$$
$$\frac{d}{dt}\left[\frac{\partial\mathcal{L}}{\partial\dot{X}}\right] = (M+m)\ddot{X} - mL\ddot{\theta}cos\theta + mL\dot{\theta}^2sin\theta$$
$$\frac{\partial\mathcal{L}}{\partial X} = 0$$
$$\frac{d}{dt}\left[\frac{\partial\mathcal{L}}{\partial\dot{X}}\right] - \frac{\partial\mathcal{L}}{\partial X} = (M+m)\ddot{X} - mL\ddot{\theta}cos\theta + mL\dot{\theta}^2sin\theta - 0 = F_{net}$$
$$F_{net} = (M+m)\ddot{X} - mL\ddot{\theta}cos\theta + mL\dot{\theta}^2sin\theta$$

Solving for $\ddot{X}$ gives us our second equation of motion:

$$\ddot{X} = \frac{F_{net} + mL\ddot{\theta}cos\theta - mL\dot{\theta}^2 sin\theta}{M+m} \tag{2}$$

We have a formula for $\ddot{\theta}$ which is (1), substituting this in gets us:

$$F_{net} = (M+m)\ddot{X} - mL\left[\frac{\ddot{X}cos\theta + gsin\theta}{L}\right]cos\theta + mL\dot{\theta}^2 sin\theta$$
$$F_{net} = (M+m)\ddot{X} - m\left[\ddot{X}cos\theta + gsin\theta\right]cos\theta + mL\dot{\theta}^2 sin\theta$$
$$F_{net} = (M+m)\ddot{X} - m\ddot{X}cos\theta - mgsin\theta + mL\dot{\theta}^2 sin\theta$$

We factor out $\ddot{X}$ to get:

$$F_{net} + mgsin\theta - mL\dot{\theta}^2 sin\theta = \ddot{X}\left[(M+m) - mcos\theta\right]$$

Finally, we solve for $\ddot{X}$ to get:

$$\ddot{X} = \frac{F_{net} + mgsin\theta - mL\dot{\theta}^2 sin\theta}{(M+m) - mcos\theta} \tag{3}$$

We can do the same for $\ddot{\theta}$ by substituting in (2):

$$\ddot{\theta} = \left[\frac{F_{net} + mL\ddot{\theta}cos\theta - mL\dot{\theta}^2 sin\theta}{M+m}\right]\frac{cos\theta}{L} + \frac{gsin\theta}{L}$$
$$\ddot{\theta} - \frac{m\ddot{\theta}cos^2\theta}{M+m} = \frac{F_{net}cos\theta}{L(M+m)} - \frac{m\dot{\theta}^2 sin\theta cos\theta}{M+m} + \frac{gsin\theta}{L}$$

This leaves us with:

$$\ddot{\theta} = \frac{\frac{F_{net}cos\theta}{L(M+m)} - \frac{m\dot{\theta}^2 sin\theta cos\theta}{M+m} + \frac{gsin\theta}{L}}{1 - \frac{mcos^2\theta}{M+m}} \tag{4}$$

Control Theory

Below is a schematic showing how a PID control system works. The PID control system is what analyzes the data from the rotary encoder and stepper motor to decide how much the cart needs to move to stabilize the pendulum in an inverted position.

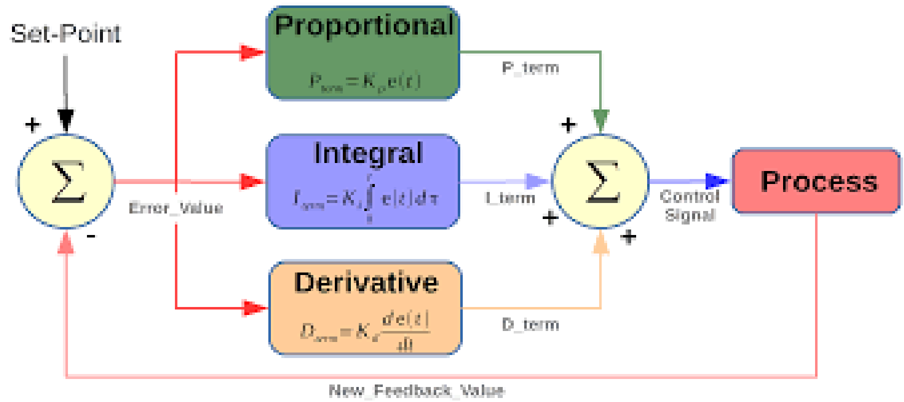

Figure 2: PID control system

Figure 2 demonstrates the underlying concept of PID control. The set point send the error data to the three components. The proportional part determines the direction the cart needs to move in order to attain the desired orientation of the pendulum. The integral component gives insight from past data to allow the system to understand how the error value has been changing over time. The derivative aspect shows the trend in how the pendulum is currently moving, which gives the system insight as to how it will need to move to either counter or better that trend. These three components combine to give a highly precise signal to send to the system.

Tuning

Tuning a PID controller is extremely important to its functionality. This means varying the coefficients for each of the three PID steps. In this project, the PID was tuned by graphing the PID data vs time and trying to smoothen the graph as much as possible. It was also tuned by observing why the pendulum would fail in certain positions, and then adjusting the coefficients based on this feedback.

4 Methods

Components

Part	Tools/Equipment	3D Printed
NEMA 17 Stepper Motor	Caliper	Rotary Encoder Mount
Stepper Motor Mount	Laptop	Two 6mm Pillow Case Bearing Mounts
Two 6mm Pillow Case Bearing	Band Saw	Pendulum
Two End Stops	Drill Press	Arduino Mount
Arduino UNO/CNC Shield/Driver	Drill Bits	End Stop Holders
2x 2pcs 1000mm Extruded Aluminum	Hacksaw	
Corner Mounts w/ T-nuts and Screws	Deburring Tool	
Three Linear Bearings	Wire Cutter	
Two 8mmx800mm Linear Motion Rods	Crimper	
Four 8mm Aluminum Mounts	Male and Female Crimp Connectors	
High-Precision Rotary Encoder	12V 2A Power Supply	
Mount for Pendulum to 6mm Shaft	Soldering Iron	
6mm Rotary Encoder Rod	Saftey Glasses	
GT2 Timing Belt/Tensioners	Allan Wrench Set	
40 Teeth GT2 Pulley	Screwdriver	
Cart Base (Aluminum Plate)	Sharpie	
Rubber Feet		
Stop switch		
Extra Fasteners		

Assembly and Testing

1. Extruded aluminum beams and corner brackets are used to create an rectangular base that is 1000 mm x 240 mm, with a 240mm cross beam 216mm from one of the edges.
2. Attach the rubber feet to the bottoms of the corners of the base.
3. Mount the 8mm aluminum mounts 50mm on both sides from the middle of the cross beam, as well as on the other side on the side beam.
4. Attach two linear bearings 133mm away from each other on one 800mm chrome rod, and attach one linear bearing in the middle of the other two, but on the other 800mm chrome rod.
5. Connect the two 800mm chrome rods to the aluminum mounts.
6. Use the stepper motor mount to mount the stepper motor between the cross beam and the side beam so that the spinning wheel is in the middle of the two 800mm chrome rods.
7. Mount the 40 teeth GT2 pulley between the two chrome rods on the side opposite to the side with the cross beam.
8. Attach the car base to the linear bearings, cutting it so that the cart just barely reaches over the extruded aluminum base.
9. Mount the rotary encoder to the plate, and mount the 6mm pendulum mount to the edge of the cart on a side that is parallel to the length of the base.
10. Attach the 6mm rotary encoder rod to the pendulum mount and the rotary encoder, on the other side of the pendulum mount is where the pendulum will be attached.
11. Attach the GT2 belt mount to the bottom of the cart, and then attach the GT2 belt and tensioners to the stepper motor, the bottom of the cart, and the 40 teeth pulley.

12. Put the end stops on both ends of one of the chrome rods.
13. Attach the arduino with shield to the power supply, and then attach the arduino to the stepper motor and to a laptop.
14. Attach the stop switch to the stepper motor and power supply, program this so it completely shuts off the system immediately after being pressed.

5 Data

The data collected comes in the form of angle measurements from the rotary encoder. This data is then used in the PID control system to determine how much the stepper motor needs to drive the cart. At the fastest, the program can calculate new PID values every 500 s, meaning 2000 values per second. This PID value is converted to delay time, which is how much delay between the high and low pules used to drive the stepper motor. Since one step is completed between pulses, and lower delay time means higher speeds, as the whole step would be completed quicker. Below is an example of calculated data, showing the speeds at which values are calculated and how they change over time:

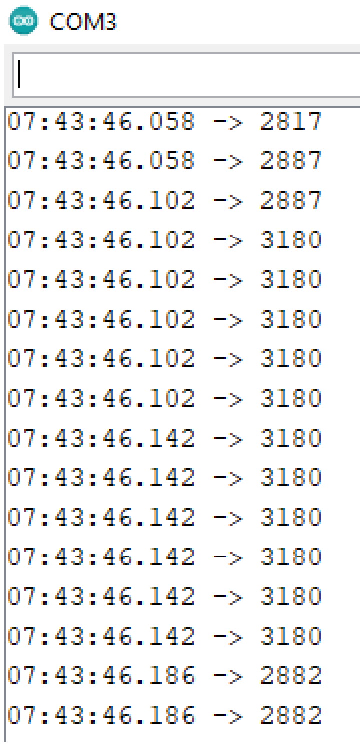

Figure 3: Delay Time Values

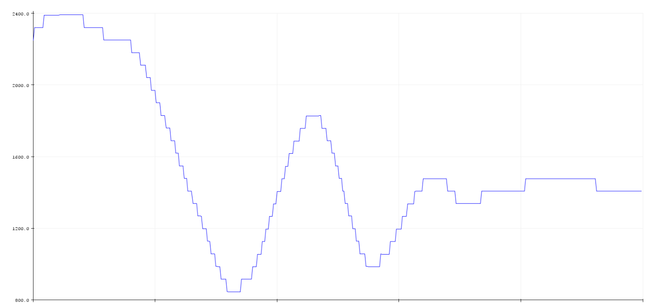

Figure 4: Result of PID calculation vs time

6 Discussion and Conclusions

By the end of the semester, a functioning inverted pendulum was achieved. While the amount of time it works for depends on the position it is started in and varies, it is able to stabilise itself in an inverted position for over 15 seconds. In order to maintain stability for longer and keep stability times from varying based on how it is started, two important things need to be done.

First, it does seem a stronger motor is needed. This is due to both to a need for higher speeds and slower speeds. When the pendulum goes further than six to eight degrees away from upright, the cart is unable to accelerate quickly enough to return the pendulum back to upright. Additionally, the cart is unable to move smoothly as slow speeds, causing the cart to abruptly stop instead of gradually slowing, which causes unwanted net force applied to the base of the pendulum. The addition of a higher torque motor would solve these problems, as well as allow for enough force for a quick swing up from the downright position to the desired upright position. In the design, the belt had to be made loose as the motor did not have enough torque to handle the tensioned belt. Due to this, the belt will sometimes become detached from the pulley connected to the motor. A stronger motor would allow for a properly tensioned belt, which would also lead to better precision and control over the carts motion.

Second, the arduino needs to keep track of how many steps the cart is away from the edges. A majority of the time, the pendulum falls and becomes unstable due to the cart running out of track. Therefore, when the cart is near the edge, a strong acceleration needs to be applied to force the pendulum to the opposite side, and thus make the cart drive away from the edge. To do this, microswitches need to be applied to the ends of the rails. The cart can then calibrate itself by seeing how many steps it is from one end to the other. This will also allow for the cart to begin in the exact center every time.

A few additions would also make the project slightly better and safer. The first being the addition of a stop button that is connected between the power supply and the motor. This would allow for a quick shut off of power to the motor in case something is going wrong. Next, mounting the arduino, stop button, and power supply to the base of the pendulum would make the project look more professional. The cart base is also wider than needed, so cutting this down to the necessary size would reduce the mass of the base and therefore allow for higher accelerations to be applied to the base of the pendulum. Finally, the linear bearings are functional but not high quality, so upgrading these would help reduce the friction between the rails and the cart base.

A future student could add these features to the pendulum, as well as try to make it a double inverted pendulum. Another year of hard work could make this project even more special, so anyone interested in learning how to work with integrated circuits and control theory to gain meaningful information about physical systems should consider building off of this existing project. After several years and several bright capstone students working on this pendulum, who knows what amazing things can be achieved.

7 Bibliography

References

[1] J.R. Taylor and S.L.L.J.R. Taylor. *Classical Mechanics*. University Science Books, 2005.